PERIODIC CONTACT PROBLEM FOR A HALF-PLANE WITH ELASTIC LAPS (COVER PLATES)

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N. Kh. ARUTIUNIAN and S. M. MKHITARIAN
(Erevan)

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A periodic contact problem for an elastic half-plane with elastic laps of finite length and constant thickness is considered herein.

The solution of this problem reduces to a singular integro-differential equation with a Hilbert kernel in an interval not coincident with $(-\pi, \pi)$, which permits the determination of contact stresses along the sections where the elastic laps are fastened to the half-plane. An effective solution of this equation containing explicitly those singularities which characterized the state of stress of the elastic laps in the neighborhoods of their ends, is presented.

Insofar as we know, this is the first formulation and solution of the problem mentioned.

1. Formulation of the problem. Derivation of the fundamental equation and its solution. Let a half-plane be reinforced at finite segments $[-a + 2nl, a + 2nl]$ $(l > a,\ n = 0, \pm 1, \pm 2, \ldots)$ by elastic fastenings duplicated periodically with period $2l$ in the form of welded (or glued) elastic laps of constant sufficiently small thickness h (Fig. 1). The purpose of the investigation is to determine the law of contact stress distribution along the segments where the elastic laps are fastened to the elastic half-plane when concentrated forces P directed along their axis are applied to one of the ends of the laps. As in [1], we shall assume that the bending

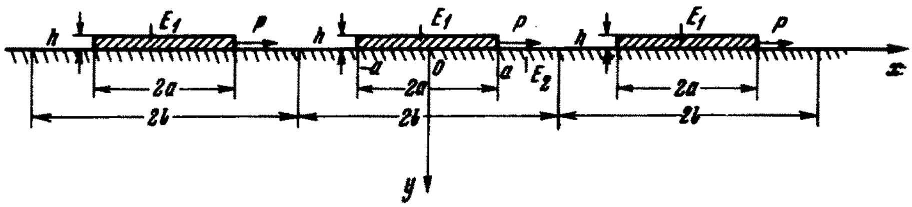

Fig. 1

stiffness of the laps is negligibly small because of the smallness of the thickness h, and hence, the normal pressure of the laps on the half-plane can be neglected. In other words, we assume that only tangential contact stresses act on the laps, i.e. they are in a uniaxial state of stress.

Let us utilize the following system of notation: we denote the displacements and strains in the laps by the subscript 1, and in the half-plane by the subscript 2, and analogously for the notation of the physical constants of the materials of the laps and of the half-plane.

Since the law of contact stress distribution under the laps is the same because of the periodic nature of the problem, the considerations can be limited to one of them, say that for which $n = 0$. Forming the equilibrium equation of this lap, and then utilizing Hooke's law, we can establish the relationship

$$\varepsilon_x^{(1)} = \frac{du^{(1)}}{dx} = \frac{1}{hE_1} \int_{-a}^{x} \tau^{(1)}(\xi)\, d\xi \tag{1.1}$$

Here E_1 is the elastic modulus of the lap material, $u^{(1)}(x)$ are the horizontal displacements of points connecting the lap to the elastic half-plane, i.e. points of the segment $[-a, a]$, and $\tau^{(1)}(x)$ are the tangential stresses acting on the lap along the line connecting it to the half-plane.

On the other hand, on the basis of the reciprocity law it is known [2] that the horizontal displacements $u^{(2)}(x)$ of boundary points of the elastic half-plane, caused by the tangential contact stresses of intensity $\tau^{(2)}(x)$, distributed over the segment $[-a, a]$ and repeated periodically with period $2l$, are determined by the formula

$$u^{(2)}(x) = \frac{2(1-\nu^2)}{\pi E_2} \int_{-a}^{a} \ln \frac{1}{2\,|\sin[\pi(x-\xi)/2l]\,|}\, \tau^{(2)}(\xi)\, d\xi$$

where ν is Poisson's ratio, and E_2 is the elastic modulus of the half-plane.

We hence obtain

$$\varepsilon_x^{(2)} = \frac{du^{(2)}}{dx} = \frac{1-\nu^2}{E_2 l} \int_{-a}^{a} \operatorname{ctg} \frac{\pi(\xi - x)}{2l}\, \tau^{(2)}(\xi)\, d\xi \tag{1.2}$$

The integral is understood here in the Cauchy principal value sense.

The condition $\qquad u^{(1)}(x) = u^{(2)}(x) \qquad (y = 0, \quad -a \leqslant x \leqslant a) \qquad (1.3)$

should be satisfied on the contact section $[-a, a]$ between the elastic lap and the half-plane, or if differentiation is performed, the condition

$$\frac{du^{(1)}(x)}{dx} = \frac{du^{(2)}(x)}{dx} \qquad (y = 0, \quad -a \leqslant x \leqslant a) \tag{1.4}$$

It should be noted that conditions (1.3), (1.4) are equivalent, since the constant which appears when (1.4) is integrated is zero because the elastic laps are welded to the half-plane, and hence, they should be displaced together, as one whole.

Substituting the expressions for $\varepsilon_x^{(1)}$ and $\varepsilon_x^{(2)}$ from (1.1), (1.2) into condition (1.4), we arrive at the singular integro-differential equation

$$\frac{1}{2\pi} \int_{-a}^{a} \operatorname{ctg} \frac{\pi(\xi - x)}{2l}\, \psi(\xi)\, d\xi = -\lambda \psi(x) \tag{1.5}$$

where

$$\psi(x) = \int_{-a}^{x} \tau(s)\, ds, \qquad \tau(x) = \tau^{(1)}(x) = -\tau^{(2)}(x), \qquad \lambda = \frac{E_2 l}{2\pi(1-\nu^2) h E_1}$$

and the integral on the left should be understood in the Cauchy principal value sense.

It is easily seen from (1.1) that the function $\psi(x)$ should satisfy the conditions

$$\psi(-a) = 0, \qquad \psi(a) = P \tag{1.6}$$

Therefore, the periodic contact problem for an elastic half-plane reinforced by periodically repeated elastic laps of constant thickness h with period $2l$, reduces to the solution of the integro-differential equation (1.5) with a Hilbert kernel under the boundary conditions (1.6).

Putting

$$\frac{\pi x}{l} = t, \qquad \frac{\pi \xi}{l} = s, \qquad \frac{\pi a}{l} = \alpha, \qquad \psi\left(\frac{lt}{\pi}\right) = \varphi(t)$$

we represent the integro-differential equation (1.5) under the boundary conditions (1.6) as

$$\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\operatorname{ctg}\frac{s-t}{2}\varphi'(s)\,ds=-\lambda\varphi(t) \tag{1.7}$$

$$\varphi(-\alpha)=0,\qquad \varphi(\alpha)=P \tag{1.8}$$

The contact stress will now be determined by the formula

$$\tau(x)=\frac{\pi}{l}\varphi_t'(t)\qquad\left(t=\frac{\pi x}{l}\right) \tag{1.9}$$

Let us turn to the solution of the integro-differential equation (1.7) under the boundary conditions (1.8). To do this, let us first invert (1.7) by considering it as an integral equation of the first kind with kernel $(2\pi)^{-1}\operatorname{ctg}{}^1/_2(s-t)$ and the unknown function $\varphi'(t)$. The inversion formula for this equation is known to be (*) [2, 3]

$$\varphi'(t)=\frac{\lambda}{2\pi\sqrt{\cos t-\cos\alpha}}\int_{-\alpha}^{\alpha}\frac{\sqrt{\cos s-\cos\alpha}\,\varphi(s)\,ds}{\sin{}^1/_2(s-t)}+\frac{A\sqrt{2}\cos{}^1/_2t}{\sqrt{\cos t-\cos\alpha}} \tag{1.10}$$

To determine the unknown constant A we integrate both sides of (1.10) and hence obtain

$$\varphi(t)=\frac{\lambda}{2\pi}\int_{-\alpha}^{\alpha}\sqrt{\cos s-\cos\alpha}\,\varphi(s)\,ds\int\frac{dt}{\sqrt{\cos t-\cos\alpha}\sin{}^1/_2(s-t)}+$$
$$+2A\arcsin\frac{\sin{}^1/_2t}{\sin{}^1/_2\alpha}+C$$

Let us evaluate the inner integral by first representing it as

$$J(t,s)=\int\frac{dt}{\sqrt{\cos t-\cos\alpha}\sin{}^1/_2(s-t)}=$$
$$=\frac{1}{\sqrt{2}\cos(s/2)}\int\frac{dt}{\sqrt{\cos^2(t/2)-\cos^2(\alpha/2)}\,[\operatorname{tg}(s/2)-\operatorname{tg}(t/2)]\cos(t/2)}$$

Here putting

$$u=\frac{\operatorname{tg}{}^1/_2t}{\operatorname{tg}{}^1/_2\alpha},\qquad y=\frac{\operatorname{tg}{}^1/_2s}{\operatorname{tg}{}^1/_2\alpha}$$

we obtain after elementary manipulation

$$J(t,s)=\frac{\sqrt{2}}{\sin{}^1/_2\alpha\cos{}^1/_2s}\int\frac{du}{\sqrt{1-u^2}\,(y-u)}$$

But the expression for the last integral has been presented in [4]. Utilizing this expression, and returning to the previous variables, we find

$$J(t,s)=\frac{1}{\sqrt{\cos s-\cos\alpha}}\ln\frac{2\cos{}^1/_2t\cos{}^1/_2s-2(\cos{}^1/_2\alpha)^2\cos{}^1/_2(t-s)+\vartheta(t,s)}{2\cos{}^1/_2t\cos{}^1/_2s-2(\cos{}^1/_2\alpha)^2\cos{}^1/_2(t-s)-\vartheta(t,s)}$$

$$\vartheta(t,s)=\sqrt{(\cos t-\cos\alpha)(\cos s-\cos\alpha)} \tag{1.11}$$

Keeping in mind the expression found for $J(t,s)$, we obtain

$$\varphi(t)=\frac{\lambda}{2\pi}\int_{-\alpha}^{\alpha}K(t,s)\varphi(s)\,ds+2A\arcsin\frac{\sin{}^1/_2t}{\sin{}^1/_2\alpha}+C \tag{1.12}$$

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*) In [3] the solution of the mentioned integral equation is reduced to the solution of some Riemann boundary value problem. An error was admitted there in constructing the canonical solution of this problem, which reduces finally to the nonintegral term in the inversion formula containing $\sin{}^1/_2t$ rather than $\cos{}^1/_2t$.

$$K(t, s) = J(t, s)\sqrt{\cos s - \cos\alpha} \tag{1.13}$$

It is easy to see that $K(\alpha, s) \equiv K(-\alpha, s) \equiv 0$, $(-\alpha \leqslant s \leqslant \alpha)$. This latter condition together with the boundary conditions (1.8) permit determination of the unknown constants from (1.12) $A = {}^1/_2 P / \pi, \quad C = {}^1/_2 P$

We simultaneously find the following result: the solution of the integro-differential equation (1.7) under the boundary conditions (1.8) is equivalent to the solution of linear second-order Fredholm integral equation

$$\varphi(t) = \frac{\lambda}{2\pi}\int_{-\alpha}^{\alpha} K(t, s)\varphi,(s)\,ds + \frac{P}{\pi}\arcsin\frac{\sin{}^1/_2 t}{\sin{}^1/_2\alpha} + \frac{P}{2} \tag{1.14}$$

with kernel $K(t, s)$ being expressed by formulas (1.13) and (1.11).

It follows directly from the results of [4] (Sect. 3), that the integral operator generated by the kernel $K(t, s)$ $(-\alpha \leqslant t, s \leqslant \alpha)$ and defined by (1.13) and (1.11) is completely continuous in the space $L_2(-\alpha, \alpha)$, and transforms elements of this space again into elements of the same space. Moreover, it is a Hilbert-Schmidt operator. This important circumstance permits obtaining the solution of the integral equation (1.14) by the known method of successive approximation. However, it must be noted that those singularities which are exceptionally important to clarification of the physical picture of the state of stress of the elastic laps in the neighborhoods of their ends, and which characterize the mechanical essence of the problem under consideration in a known sense, are not manifest in such a method of solving the integro-differential equation (1.7).

In order to represent explicitly those singularities which are inherent to the tangential contact stresses $\tau(x)$ under the elastic laps in the neighborhoods of their ends, we propose another method of solving the integro-differential equation (1.7) subject to the boundary conditions (1.8). This method affords the possibility not only of representing the singularities in the neighborhood of the ends of the laps explicitly, which is most important to us, but permits, moreover, finding the values of the contact stress $\tau(x) = \pi / l\, \varphi'(t)$ $(t = \pi x / l)$ under the laps directly. Finally, this method permits the construction of approximate values of the contact stress $\tau(x)$, say the nth approximation, by passing the determination of the previous approximations and estimating this approximation.

Now, let us turn to an exposition of the method. The contact stress $\tau(x)$ $(-a \leqslant x \leqslant a)$ equals zero identically for $a < |x| < l$, changes periodically with period $2l$, hence it can be represented as a Fourier series. When using the variable t this period becomes equal to 2π, therefore we can write

$$\varphi'(t) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} k\beta_k \cos kt - k\alpha_k \sin kt \qquad (-\alpha \leqslant t \leqslant \alpha) \tag{1.15}$$

Let us emphasize that the trigonometric series in the right side of (1.15) is a Fourier series expansion of the function

$$f(t) = \begin{cases} 0 & (-\pi < t < -\alpha) \\ \varphi'(t) & (-\alpha \leqslant t \leqslant \alpha) \\ 0 & (\alpha < t < \pi) \end{cases} \tag{1.16}$$

i.e.

$$f(t)=\frac{\alpha_0}{2}+\sum_{k=1}^{\infty} k\beta_k \cos kt - k\alpha_k \sin kt \qquad (-\pi<t<\pi) \qquad (1.17)$$

because of the above.

Hence,we find $\alpha_0 = P/\pi$ for the zero coefficient according to the Fourier formula and utilization of (1.15) and the boundary conditions (1.8).

Putting

$$F(t)=\int_{-\pi}^{t} f(s)\,ds \qquad (-\pi<t<\pi) \qquad (1.18)$$

and integrating both sides of (1.17) in the range $(-\pi,\ t)$ we obtain

$$F(t)=\gamma+\frac{P}{2\pi}t+\sum_{k=1}^{\infty}\alpha_k\cos kt+\beta_k\sin kt \qquad (-\pi<t<\pi) \qquad (1.19)$$

where γ is a constant of integration. The method to determine it will be indicated in the next section.

The function $F(t)$ is defined only in the interval $(-\pi,\ \pi)$. On the rest of the real axis we define it by means of a periodic continuation of the function (1.18) with period 2π . It is easily seen from (1.19) that this continuation actually reduces to a periodic continuation of the linear function $g(t)=\gamma+{}^1/_2\, Pt/\pi$ with period 2π . We shall henceforth understand the function $F(t)$ to be the function (1.18) continued in precisely this way over the whole real axis.

Taking account of (1.16) and (1.18), it is easy to note that

$$F(t)=\begin{cases} 0 & (-\pi<t\leqslant\alpha) \\ \varphi(t) & (-\alpha\leqslant t\leqslant\alpha) \\ P & (\alpha\leqslant t<\pi) \end{cases} \qquad (1.20)$$

Therefore, the expansion

$$\varphi(t)=\gamma+\frac{P}{2\pi}t+\sum_{k=1}^{\infty}a_k\cos kt+\beta_k\sin kt \qquad (-\alpha\leqslant t\leqslant\alpha) \qquad (1.21)$$

holds.

Let us expand the linear function $g(t)=\gamma+{}^1/_2 Pt/\pi$ continued periodically over the whole real axis with period 2π , in a Fourier series

$$g(t)=\gamma+\frac{P}{2\pi}t=\gamma+2P\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kt \qquad (-\pi<t<\pi)$$

Then the function $\varphi(t)$ can be represented as

$$\varphi(t)=\gamma+\sum_{k=1}^{\infty}\alpha_k\cos kt+\left[\beta_k-(-1)^k\frac{2P}{k}\right]\sin kt \qquad (-\alpha\leqslant t\leqslant\alpha) \qquad (1.22)$$

Let us note that the need for periodic continuation of the function(1.18) results from the physical nature of the problem being examined. Namely: deformations of points of the elastic lap with number zero are determined by the function $\varphi(\pi x/l)=\psi(x)$ $(-a\leqslant x\leqslant a)$ by using (1.1), and deformations of the boundary points of the elastic half-plane $-l<x<l$ by the functions $F(\pi x/l)$ $(-l<x<l)$ Since the problem is periodic, the picture of the state of strain of this half-strip should be repeated periodically with period $2l$, consequently, the function $F(\pi x/l)$ should be periodic

with period $2l$, and the function $F(t)$ should be periodic with period 2π.

Therefore the validity of the representation of the function $\varphi(t)$ in the form (1.21) is well founded.

Let us turn to the exposition of the proposed method of solving the integro-differential equation (1.7) under the boundary conditions (1.8). Let us proceed from (1.10). Taking account of the value of the constant A found above, we obtain

$$\varphi'(t) = \frac{\lambda}{2\pi\sqrt{\cos t - \cos\alpha}} \int_{-\alpha}^{\alpha} \frac{\sqrt{\cos s - \cos\alpha}\,\varphi(s)\,ds}{\sin{}^1/_2(s-t)} + \frac{P\cos{}^1/_2 t}{\pi\sqrt{2(\cos t - \cos\alpha)}}$$

$$(-\alpha < t < \alpha)$$

Substituting the exposition (1.22) for the function $\varphi(t)$ into the last expression, we find

$$\varphi'(t) = \frac{P\cos{}^1/_2 t}{\pi\sqrt{2(\cos t - \cos\alpha)}} + \frac{\lambda\Upsilon}{\sqrt{\cos t - \cos\alpha}} J_0(t) + \tag{1.23}$$

$$+\frac{\lambda}{\sqrt{\cos t - \cos\alpha}}\left\{\sum_{k=1}^{\infty}\alpha_k J_k(t) + \sum_{k=1}^{\infty}\left[\beta_k - \frac{(-1)^k 2P}{k}\right] I_k(t)\right\} \quad (-\alpha < t < \alpha)$$

where

$$J_0(t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{\sqrt{\cos s - \cos\alpha}}{\sin{}^1/_2(s-t)}\,ds \tag{1.24}$$

$$J_k(t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{\sqrt{\cos s - \cos\alpha}\cos ks}{\sin{}^1/_2(s-t)}\,ds \quad (k = 1, 2, \ldots) \tag{1.25}$$

$$I_k(t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{\sqrt{\cos s - \cos\alpha}\sin ks}{\sin{}^1/_2(s-t)}\,ds \quad (k = 1, 2, \ldots) \tag{1.26}$$

These integrals are evaluated below.

For what follows, let us first note that

$$\sqrt{\cos s - \cos\alpha} = {}^1/_2\sqrt{2}\exp(-{}^1/_2 is)\,[(e^{is} - e^{i\alpha})(e^{is} - e^{-i\alpha})]^{1/2}$$

$$\sin{}^1/_2(s-t) = -{}^1/_2 i\exp[-{}^1/_2 i(t+s)]\,(e^{is} - e^{it})$$

Then formulas (1.24)–(1.26) become

$$J_0(t) = \frac{i\exp{}^1/_2 it}{\pi\sqrt{2}}\int_{-\alpha}^{\alpha}\frac{[(e^{is} - e^{i\alpha})(e^{is} - e^{-i\alpha})]^{1/2}}{e^{is} - e^{it}}\,ds$$

$$J_k(t) = \frac{i\exp{}^1/_2 it}{2\pi\sqrt{2}}\int_{-\alpha}^{\alpha}\frac{[(e^{is} - e^{i\alpha})(e^{is} - e^{-i\alpha})]^{1/2}(e^{iks} + e^{-iks})}{e^{is} - e^{it}}\,ds \quad (k = 1, 2, \ldots)$$

$$I_k(t) = \frac{\exp{}^1/_2 it}{2\pi\sqrt{2}}\int_{-\alpha}^{\alpha}\frac{[(e^{is} - e^{i\alpha})(e^{is} - e^{-i\alpha})]^{1/2}(e^{iks} - e^{-iks})}{e^{is} - e^{it}}\,ds \quad (k = 1, 2, \ldots)$$

Let us pass from the segment $[-\alpha, \alpha]$ on the real axis to the arc $\bar{a}a$ of the unit circle, whereupon we put

$$e^{is} = \zeta, \qquad e^{it} = \sigma, \qquad e^{i\alpha} = a$$

and hence obtain

$$J_0(-i\ln\sigma) = J_0^*(\sigma) = \frac{\sqrt{2\sigma}}{2\pi}\int_{\bar{a}}^{a}\frac{[(\zeta - a)(\zeta - \bar{a})]^{1/2}d\zeta}{\zeta(\zeta - \sigma)} \tag{1.27}$$

$$J_k'(-i\ln\sigma)=J_k^*(\sigma)=\frac{\sqrt{\sigma}}{2\pi\sqrt{2}}\int_{\bar{a}}^{a}\frac{[(\zeta-a)(\zeta-\bar{a})]^{1/2}(\zeta^{k-1}+\zeta^{-k+1})}{\zeta-\sigma}\,d\zeta \tag{1.28}$$

$$I_k(-i\ln\sigma)=I_k^*(\sigma)=\frac{\sqrt{\sigma}}{2\pi i\sqrt{2}}\int_{\bar{a}}^{a}\frac{[(\zeta-a)(\zeta-\bar{a})]^{1/2}(\zeta^{k-1}-\zeta^{-k+1})}{\zeta-\sigma}\,d\zeta$$

$$(k=1,2,\ldots) \tag{1.29}$$

Proceeding to the evaluation of the integral $J_0^*(\delta)$, let us introduce the piecewise-holomorphic function

$$\Phi_0(z)=\frac{1}{2\pi i}\oint_C\frac{[(w-a)(w-\bar{a})]^{1/2}}{w(w-z)}\,dw$$

with the contour of integration C shown in Fig. 2.

The function $[(w-a)(w-\bar{a})]^{1/2}$ in this integral is a double-valued function with the branch points $w=a$, $w=\bar{a}$ located on the unit circle with center at the origin. As it is easy to see, a single-valued analytic branch of this function can be selected in the plane slit along the arc $\bar{a}a$ of the unit circle. Let us select the branch for which the radical is taken with the plus sign. Henceforth we shall understand $[(w-a)(w-\bar{a})]^{1/2}$ to be precisely this branch. Then the function

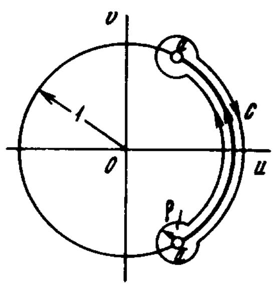

Fig. 2

$$f(w)=w^{-1}[(w-a)(w-\bar{a})]^{1/2}$$

can be represented for $|w|>1$ as

$$f(w)=1+O(w^{-1})$$

In order to clarify the structure of the function $f(w)$ at the origin, let us put

$$w-a=(a-w)e^{\pi i},\quad w-\bar{a}=(\bar{a}-w)e^{\pi i}$$

Therefore

$$f(w)=-w^{-1}(a-w)^{1/2}(\bar{a}-w)^{1/2}$$

Hence, by using the expression for binomial series, we find that in the neighborhood of the origin

$$f(w)=-w^{-1}+O(1)$$

i.e. the function $f(w)$ has a first order pole at the point $w=0$.

Therefore, the function $f(w)$ is holomorphic in the whole plane slit along the arc $\bar{a}a$, including the point $w=\infty$ (where it has a zero order pole with principal part 1), except the point $w=0$ (which is a first order pole with principal part $-w^{-1}$).

Furthermore, let us use the following result [5]. Let the function $f(w)$ be holomorphic in the domain D, which is an infinite part of the plane consisting of points located outside the closed contour C, with the exception, perhaps, of the finite points $a_1,\ldots,a_n$ of this domain, and also the point $w=\infty$ where it can have a pole with the principal parts $G_1(w),\ldots,G_n(w)$, $G_\infty(w)$ (the pole can be of zero order at the point $w=\infty$).

In this case the formula

$$\frac{1}{2\pi i}\oint_C\frac{f(w)\,dw}{w-z}=f(z)-G_1(z)-\ldots-G_n(z)-G_\infty(z)\quad(z\in D) \tag{1.30}$$

which is the Cauchy formula for an infinite domain D holds.

Applying the last formula to the function $f(w)=w^{-1}[(w-a)(w-\bar{a})]^{1/2}$ we obtain

$$\Phi_0(z) = \frac{1}{2\pi i} \oint_C \frac{[(w-a)(w-\bar{a})]^{1/2}}{w(w-z)}\, dw = \frac{[(z-a)(z-\bar{a})]^{1/2}}{z} + \frac{1}{z} - 1 \qquad (1.31)$$

Let us shrink the contour C to the slit $\bar{a}a$ along the arc of the unit circle. Let us first find the limit values of the radical on the inner and outer edges of this slit. These values are calculated quite simply in the case of a slit made along some segment of the real axis. The case of a slit along the arc of a unit circle can be reduced to the case mentioned. This is done by mapping the unit circle conformally on the upper half-plane. The mapping function is

$$w_1 = i\,\frac{1-w}{1+w}$$

For points of the unit circle $w = \zeta = e^{is}$ $(-\pi \leqslant s \leqslant \pi)$. These points are transformed into points on the real axis by the formula

$$w_1 = u_1 = \operatorname{tg} {}^1/_2 s$$

Hence, it is seen that a slit along the arc $\bar{a}a$ of the unit circle in the complex w plane passes into a slit along the finite segment $[-\operatorname{tg} {}^1/_2\, \alpha,\ \operatorname{tg} {}^1/_2\, \alpha]$ on the real axis on the complex w_1 plane under the conformal mapping mentioned. We find simultaneously that

$$w - a = -\frac{1+a}{1+w_1}\left(w_1 - \operatorname{tg}\frac{\alpha}{2}\right), \qquad w - \bar{a} = -\frac{1+\bar{a}}{1+w_1}\left(w_1 + \operatorname{tg}\frac{\alpha}{2}\right) \qquad (1.32)$$

Assuming $w_1 \to u_1$, we will consider that on the upper edge of the slit

$$w_1 - \operatorname{tg} {}^1/_2\alpha \to (\operatorname{tg} {}^1/_2\alpha - u_1)e^{\pi i}, \quad w_1 + \operatorname{tg} {}^1/_2\alpha \to \operatorname{tg} {}^1/_2\alpha + u_1$$

and on the lower edge of the slit

$$w_1 - \operatorname{tg} {}^1/_2\alpha \to (\operatorname{tg} {}^1/_2\alpha - u_1)e^{-\pi i}, \quad w_1 + \operatorname{tg} {}^1/_2\alpha \to \operatorname{tg} {}^1/_2\alpha + u_1$$

in the segment $[-\operatorname{tg} {}^1/_2\, \alpha,\ \operatorname{tg} {}^1/_2\alpha]$.

The validity of these relationships is easily seen if the changes in the arguments of the complex numbers $w_1 - \operatorname{tg} {}^1/_2\alpha$ and $w_1 + \operatorname{tg} {}^1/_2\alpha$ are followed as $w_1 \to u_1$.

Taking into account the above, we establish by using the transformation formulas (1.32) that the radical $[(w-a)(w-\bar{a})]^{1/2}$ taken on the respective values

$$i[(a-\zeta)(\zeta-\bar{a})]^{1/2}, \quad -i[(a-\zeta)(\zeta-\bar{a})]^{1/2}$$

on the inner and outer edges of the slit along the arc $\bar{a}a$ of the unit circle, or keeping in mind that $i = +\sqrt{-1}$, the values

$$-[(\zeta-a)(\zeta-\bar{a})]^{1/2}, \ [(\zeta-a)(\zeta-\bar{a})]^{1/2} \quad (\zeta \in \bar{a}a) \qquad (1.33)$$

Taking into account the values (1.33), let us shrink the contour C in (1.31) to the slit $\bar{a}a$ (*). We obtain

$$\frac{1}{2\pi i}\int_{\bar{a}}^{a} \frac{-2[(\zeta-a)(\zeta-\bar{a})]^{1/2}d\zeta}{\zeta(\zeta-z)} = \frac{[(z-a)(z-\bar{a})]^{1/2}}{z} + \frac{1}{z} - 1$$

Hence, applying the Sokhotskii-Plemelj formula, and again utilizing the values (1.33), we find that to the left and right of the slit as $z \to \sigma$, i.e. as the point z tends to the point σ of the inner or outer edges of the slit, the following relationship holds

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*) Since the integrand is on the order of $O((w-\bar{a})^{1/2})$ and $O((w-a)^{1/2})$, respectively, on the ends $\bar{a}$ and a of the slit, the integrals taken over small circles will tend to zero as $\rho \to 0$.

$$\frac{1}{\pi i}\int_{\bar{a}}^{a}\frac{[(\zeta-a)(\zeta-\bar{a})]^{1/2}d\zeta}{\zeta(\zeta-\sigma)}=1-\frac{1}{\sigma}$$

Therefore, the integral $J_0^*(\sigma)$ defined by (1.27) is

$$J_0^*(\sigma)={}^1/_2 i\sqrt{2\sigma}(1-\sigma^{-1})$$

Putting $\sigma=e^{it}$ we obtain the expression for the integral $J_0(t)$

$$J_0(t)=-\sqrt{2}\sin {}^1/_2 t \tag{1.34}$$

Let us turn to the evaluation of the integrals (1.28) and (1.29). Let us introduce the notation

$$X_k(\sigma)=\frac{1}{2\pi}\int_{\bar{a}}^{a}\frac{[(\zeta-a)(\zeta-\bar{a})]^{1/2}\zeta^{k-1}}{\zeta-\sigma}\,d\zeta \qquad (k=1,2,\ldots)$$

We will then have

$$\begin{aligned} J_k^*(\sigma)&={}^1/_2\sqrt{2\sigma}\,[X_k(\sigma)+X_{-k}(\sigma)] \\ I_k^*(\sigma)&=-{}^1/_2 i\sqrt{2\sigma}\,[X_k(\sigma)-X_{-k}(\sigma)]\end{aligned} \qquad (k=1,2,\ldots) \tag{1.35}$$

Let us consider the piecewise-holomorphic functions

$$\begin{aligned}\Phi_k(z)&=\frac{1}{2\pi i}\oint_C\frac{[(w-a)(w-\bar{a})]^{1/2}w^{k-1}}{w-z}\,dw \\ \Phi_{-k}(z)&=\frac{1}{2\pi i}\oint_C\frac{[(w-a)(w-\bar{a})]^{1/2}}{w^{k+1}(w-z)}\,dw\end{aligned} \qquad (k=1,2\ldots.)$$

where the contour of integration C is as before.

Let us investigate the analytic properties of the functions

$$f_k(w)=[(w-a)(w-\bar{a})]^{1/2}w^{k-1} \quad (k=1,2\ldots)$$

where, as before, the radical is taken with the positive sign. It is easy to see that in the neighborhood of the origin these functions are holomorphic. In order to clarify the structure of these functions for large w, we represent them as

$$f_k(w)=w^k\left(1-\frac{a}{w}\right)^{1/2}\left(1-\frac{\bar{a}}{w}\right)^{1/2}$$

It is easy to establish, by multiplying the binomial series which are power series expansions of the square roots entering here, that for $|w|>1$ these functions admit the representation

$$f_k(w)=\sum_{n=0}^{\infty}(-1)^n\frac{C_n}{w^{n-k}}=\sum_{n=0}^{k}(-1)^nC_nw^{k-n}+H(w)$$

Here $H(w)$ is a holomorphic function at ∞

$$C_n=\sum_{p=0}^{n}C_{1/2}^{(p)}C_{1/2}^{(n-p)}a^p\bar{a}^{n-p}$$

It is easy to show that

$$C_n=\bar{C}_n=\sum_{p=0}^{n}C_{1/2}^{(p)}C_{1/2}^{(n-p)}\cos(2p-n)\alpha$$

Therefore, the functions $f_k(w)$ $(k=1,2\ldots)$ are holomorphic in the whole plane slit along the arc $\bar{a}a$ of the unit circle, except at the infinitely remote point which is a pole of order k with the principal parts

$$G_\infty^{(k)}(w)=\sum_{n=0}^{k}(-1)^nC_nw^{k-n} \qquad (k=1,2,\ldots)$$

Similarly, it is easily shown that the functions

$$f_{-k}(w) = \frac{[(w-a)(w-\bar{a})]^{1/2}}{w^{k+1}} \qquad (k=1,2,\ldots)$$

are holomorphic in the whole plane of the variable w slit along the arc $\bar{a}a$, with the exception of the point $w=0$, at which they have a pole of order k with the principal parts

$$G_0^{(k)}(w) = \sum_{n=0}^{k} \frac{(-1)^{n+1}C_n}{w^{k-n+1}}$$

Taking into account the mentioned analytical properties of the functions $f_k(w)$ and $f_{-k}(w)$ ($k=1,2,\ldots$) and applying the Cauchy formula (1.30), we find by a method completely analogous to that expounded above for the calculation of the integral $J_0^*(\sigma)$ that

$$X_k(\sigma) = \frac{i}{2}\sum_{n=0}^{k}(-1)^n C_n \sigma^{k-n}, \quad X_{-k}(\sigma) = \frac{i}{2}\sum_{n=0}^{k}(-1)^{n+1}C_n\sigma^{n-k-1} \qquad (k=1,2,\ldots)$$

Using the last formulas, we find expressions for the integrals $J_k^*(\sigma)$ and $I_k^*(\sigma)$ from (1.35), and then the following expressions for the integrals $J_k(t)$ and $I_k(t)$ from (1.25) and (1.26):

$$J_k(t) = -\frac{1}{\sqrt{2}}\sum_{n=0}^{k}(-1)^n C_n \sin(k-n+1/2)t$$

$$(k=1,2,\ldots) \qquad (1.36)$$

$$I_k(t) = \frac{1}{\sqrt{2}}\sum_{n=0}^{k}(-1)^n C_n \cos(k-n+1/2)t$$

$$C_n = \sum_{p=0}^{n} C_{1/2}^{(p)} C_{1/2}^{(n-p)} \cos(2p-n)\alpha \qquad (1.37)$$

Before turning to the solution of the integro-differential equation (1.7) under the boundary conditions (1.8), let us evaluate the coefficients C_n in the formulas presented above.

We find directly from (1.37) $\quad C_0 = 1, \; C_1 = \cos\alpha$

The remaining coefficients C_n are easily evaluated as follows. Let us consider the function

$$h(w) = [(w-a)(w-\bar{a})]^{1/2}$$

where the radical is taken with the positive sign. The single-valued analytic branch of this function is thereby selected in the plane slit along the arc $\bar{a}a$ of the unit circle. It is easy to show, as above, that

$$h(w) = -\sum_{n=0}^{\infty}(-1)^n C_n w^n \qquad (|w|<1) \qquad (1.38)$$

$$h(w) = w\sum_{n=0}^{\infty}(-1)^n \frac{C_n}{w^n} \qquad (|w|>1) \qquad (1.39)$$

Let the counter-clockwise direction be considered the positive direction on the unit circle. Let $h_+(\sigma)$ denote the limit values of $h(w)$ when the point w tends to the point σ on the unit circle from the left, and $h_-(\sigma)$ the limit values of this function when the point w tends to the same point σ from the right. Let us form the difference $h_+(\sigma) - h_-(\sigma)$. Since the function $h(w)$ is analytic on the whole unit circle except the slit along the arc $\bar{a}a$, then this difference will vanish for all points σ of the unit circle not belonging to the slit. Let us find the values of this difference on the slit along the arc $\bar{a}a$. To do this, let us utilize the values (1.33) of the radical on the inner and outer edges of the slit. We obtain that

$$h_+(\sigma) - h_-(\sigma) = 2\sqrt{(\sigma - a)(\sigma - \bar{a})} \qquad (\sigma \in \bar{a}a)$$

Therefore

$$h_+(\sigma) - h_-(\sigma) = \begin{cases} 0, & \sigma \bar{\in} \bar{a}a \\ -2\sqrt{(\sigma - a)(\sigma - \bar{a})}, & \sigma \in \bar{a}a \end{cases}$$

On the other hand, we obtain from the expansions (1.38), (1.39)

$$h_+(\sigma) - h_-(\sigma) = -\left[\sum_{n=0}^{\infty} (-1)^n C_n \sigma^n + \sum_{n=0}^{\infty} \frac{(-1)^n C_n}{\sigma^{n-1}}\right]$$

Comparing the expressions obtained for $h_+(\sigma) - h_-(\sigma)$ by the two methods, we discover that

$$\sum_{n=0}^{\infty} (-1)^n C_n \sigma^n + \sum_{n=0}^{\infty} (-1)^n \sigma^{1-n} = \begin{cases} 2\sqrt{(\sigma - a)(\sigma - \bar{a})}, & \sigma \in \bar{a}a \\ 0, & \sigma \bar{\in} \bar{a}a \end{cases}$$

Putting $\sigma = e^{it}$, $a = e^{i\alpha}$ we will have

$$\sum_{k=-\infty}^{\infty} d_k e^{ikt} = q(t) = \begin{cases} 2\exp(^1/_2 it)\sqrt{2(\cos t - \cos\alpha)} & (-\alpha \leqslant t \leqslant \alpha) \\ 0 & (-\pi + \alpha < t < \pi - \alpha \end{cases} \qquad (1.40)$$

where we have used the following notation:

$$d_0 = C_0 - C_1, \qquad d_1 = C_0 - C_1$$
$$d_k = (-1)^k C_k \quad (k = 2,3\ldots), \quad d_{-k}(-1)^{k+1}C_{k+1} \quad (k = 1,2,\ldots) \qquad (1.41)$$

Furthermore, let us expand the function $q(t)$ in a Fourier series

$$q(t) = \sum_{k=-\infty}^{\infty} q_k e^{ikt} \qquad (1.42)$$

For the Fourier coefficients

$$q_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} q(t) e^{-ikt} dt$$

we obtain the expressions

$$q_k = \frac{P_{k-2}(\cos\alpha) - P_k(\cos\alpha)}{2k-1} \qquad (k = 0, \pm 1, \pm 2,\ldots)$$

where $P_k(\cos\alpha)$ are Legendre polynomials.

Substituting its Fourier series expansion (1.42) into (1.40) instead of $q(t)$ comparing coefficients of e^{ikt}, and utilizing the notation (1.41), we find

$$C_k = (-1)^k \frac{P_{k-2}(\cos\alpha) - P_k(\cos\alpha)}{2k-1} \qquad (k = 2, 3,\ldots)$$

Let us note that the coefficients C_0 and C_1 are not themselves determined by this means; only their difference has been determined.

We therefore have

$$C_0 = 1, \quad C_1 = \cos\alpha, \quad C_n = (-1)^n \frac{P_{n-2}(\cos\alpha) - P_n(\cos\alpha)}{2n-1} \qquad (n = 2, 3,\ldots) \qquad (1.43)$$

Now the solution (1.23) of the integro-differential equation (1.7) with the boundary conditions (1.8) can be represented after the expressions (1.34) and (1.36) have been substituted for the integrals $J_0(t)$, $J_k(t)$ and $I_k(t)$, as

$$\varphi'(t) = \frac{P\cos{}^1/_2 t}{\pi\sqrt{2(\cos t - \cos\alpha)}} - \frac{2\lambda\gamma\sin{}^1/_2 t}{\sqrt{2(\cos t - \cos\alpha)}} -$$

$$
-\frac{\lambda}{\sqrt{2(\cos t-\cos\alpha)}}\left\{\sum_{k=1}^{\infty}\alpha_k\sum_{n=0}^{k}(-1)^n C_n\sin(k-n+{}^1/_2)\,t-\right.
$$
$$
\left.-\sum_{k=1}^{\infty}\left[\beta_k-(-1)^k\frac{2P}{k}\right]\sum_{n=0}^{k}(-1)^n C_n\cos(k-n+{}^1/_2)\,t\right\}\quad(-\alpha<t<\alpha)\quad(1.44)
$$

where the coefficients C_n are defined by (1.43).

According to (1.9) we finally obtain the following formula for the contact stress $\tau(x)$ under the elastic laps:

$$
\tau(x)=\frac{\pi}{l}\varphi'\left(\frac{\pi x}{l}\right)=\frac{P\cos(\pi x/2l)}{l\sqrt{2(\cos(\pi x/l)-\cos(\pi a/l))}}-
$$
$$
-\frac{2\pi\lambda\gamma\sin(\pi x/2l)}{l\sqrt{2(\cos(\pi x/l)-\cos(\pi a/l))}}-\frac{\lambda\pi}{l\sqrt{2(\cos(\pi x/l)-\cos(\pi a/l))}}\times
$$
$$
\times\left\{\sum_{k=1}^{\infty}\alpha_k\sum_{n=0}^{k}(-1)^n C_n\sin\left[\left(k-n+\frac{1}{2}\right)\frac{\pi x}{l}\right]-\right.\qquad(1.45)
$$
$$
\left.-\sum_{k=1}^{\infty}\left[\beta_k-(-1)^k\frac{2P}{k}\right]\sum_{n=0}^{k}(-1)^n C_n\cos\left[\left(k-n+\frac{1}{2}\right)\frac{\pi x}{l}\right]\right\}\quad(-\alpha<x<\alpha)
$$

Therefore, the law for contact stress distribution under the elastic laps glued to an elastic half-plane and repeated periodically with period $2l$ is determined by (1.45) if the coefficients α_k and β_k are known. As will be proved in Sect. 3, those singularities which characterize the state of stress of the laps near their ends are explicitly extracted out in this formula. It will be proved in Sect. 2 that the definition of the coefficients α_k and β_k reduces to the solution of two separate infinite systems of linear algebraic equations with bounded forcing terms. It will be shown there that these infinite systems of linear equations will be completely regular for

$$
\frac{\lambda}{\sin\alpha}<\frac{1}{25},\qquad \lambda=\frac{E_2 l}{2\pi(1-\nu^2)hE_1}
$$

and quasi-completely regular for $\lambda/\sin\alpha\geqslant{}^1/_{25}$. Therefore, it is possible to rely on the theory of regular and quasi-regular infinite systems of linear equations with bounded forcing terms, and to assert that the coefficients α_k and β_k can be determined to any required accuracy.

Let us note that when $\lambda=0$, the elastic laps are replaced by rigid ones, i.e. by stamps, and (1.45) reduces to the known formula from [2].

2. Derivation and investigation of the infinite systems of linear equations. Let us return to (1.44) in order to derive the infinite systems of linear equations in the unknown coefficients α_k and β_k. Substituting its Fourier series expansion (1.15) in place of $\varphi'(t)$ in this formula, and keeping in mind that $\alpha_0=P/\pi$, we hence obtain

$$
\frac{P}{2\pi}+\sum_{m=1}^{\infty}m(\beta_m\cos mt-\alpha_m\sin mt)=\frac{P\cos{}^1/_2t}{\pi\sqrt{2(\cos t-\cos\alpha)}}-\frac{2\lambda\gamma\sin{}^1/_2t}{\sqrt{2(\cos t-\cos\alpha)}}-
$$
$$
-\frac{\lambda}{\sqrt{2(\cos t-\cos\alpha)}}\left\{\sum_{k=1}^{\infty}\alpha_k\sum_{n=0}^{k}(-1)^n C_n\sin(k-n+{}^1/_2)\,t-\right.
$$
$$
\left.-\sum_{k=1}^{\infty}\left[\beta_k-(-1)^k\frac{2P}{k}\right]\sum_{n=0}^{k}(-1)^n C_n\cos(k-n+{}^1/_2)\,t\right\}\quad(-\alpha<t<\alpha)\quad(2.1)
$$

Now, let us note that the sum of the series on the left side of the last equality should vanish identically in the intervals $(-\pi, \alpha)$ and (α, π), as is obvious from (1.16) and (1.17). Therefore, for the Fourier series expansion of the whole right side of (2.1) defined only in the interval $(-\alpha, \alpha)$, it must be continued in the remainder of the interval $(-\pi, \pi)$ by a function identically zero. Taking this fact into account, let us introduce the functions

$$g_p(t) = \begin{cases} 0, & -\pi \leqslant t < -\alpha \\ \dfrac{\cos(p+{}^1/_2)t}{\sqrt{2(\cos t - \cos\alpha)}}, & -\alpha \leqslant t \leqslant \alpha \\ 0, & \alpha < t \leqslant \pi \end{cases} \qquad (p = 0, 1, 2, \ldots) \tag{2.2}$$

$$h_p(t) = \begin{cases} 0, & -\pi \leqslant t < -\alpha \\ \dfrac{\sin(p+{}^1/_2)t}{\sqrt{2(\cos t - \cos\alpha)}}, & -\alpha \leqslant t \leqslant \alpha \\ 0, & \alpha < t \leqslant \pi \end{cases} \qquad (p = 0, 1, 2, \ldots) \tag{2.3}$$

continued periodically with period 2π over the whole real axis.

By utilizing these functions (2.1) can be written as

$$\frac{P}{2\pi} + \sum_{m=1}^{\infty} m(\beta_m \cos mt - \alpha_m \sin mt) = \frac{P\cos{}^1/_2 t}{\pi\sqrt{2(\cos t - \cos\alpha)}} -$$

$$- \frac{2\lambda\gamma \sin{}^1/_2 t}{\sqrt{2(\cos t - \cos\alpha)}} - \lambda\left\{\sum_{k=1}^{\infty} \alpha_k \sum_{n=0}^{k} (-1)^n C_n h_{k-n}(t) - \right.$$

$$\left. - \sum_{k=1}^{\infty}\left[\beta_k - (-1)^k \frac{2P}{k}\right] \sum_{n=0}^{k} (-1)^n C_n g_{k-n}(t)\right\} \qquad (-\pi \leqslant t \leqslant \pi) \tag{2.4}$$

Furthermore, the functions $g_p(t)$ and $h_p(t)$, which are even and odd respectively, are expanded in the Fourier series

$$g_p(t) = \frac{C_0^{(p)}}{2} + \sum_{m=1}^{\infty} C_m^{(p)} \cos mt, \qquad h_p(t) = \sum_{m=1}^{\infty} D_m^{(p)} \sin mt \qquad (-\pi \leqslant t \leqslant \pi)$$

whose coefficients are defined, according to (2.2) and (2.3) by

$$C_m^{(p)} = \frac{2}{\pi}\int_0^{\alpha} \frac{\cos(p+{}^1/_2)t \cos mt}{\sqrt{2(\cos t - \cos\alpha)}}\, dt \qquad (m = 0, 1, 2, \ldots)$$

$$D_m^{(p)} = \frac{2}{\pi}\int_0^{\alpha} \frac{\sin(p+{}^1/_2)t \sin mt}{\sqrt{2(\cos t - \cos\alpha)}}\, dt \qquad (m = 1, 2, \ldots)$$

Utilizing the formula [6]

$$P_\nu(\cos\alpha) = \frac{2}{\pi}\int_0^{\alpha} \frac{\cos(\nu+{}^1/_2)t}{\sqrt{2(\cos t - \cos\alpha)}}\, dt \qquad (0 < \alpha < \pi)$$

for the Legendre function of the first kind of order ν we obtain at once

$$C_m^{(p)} = {}^1/_2\,[P_{p+m}(\cos\alpha) + P_{p-m}(\cos\alpha)], \quad D_m^{(p)} = {}^1/_2\,[P_{p-m}(\cos\alpha) - P_{p+m}(\cos\alpha)]$$

Therefore

$$g_p(t) = \frac{P_p(\cos\alpha)}{2} + \sum_{m=1}^{\infty} \frac{P_{p+m}(\cos\alpha) + P_{p-m}(\cos\alpha)}{2} \cos mt \quad (p = 0, 1, 2, \ldots) \tag{2.5}$$

$$h_p(t) = \sum_{m=1}^{\infty} \frac{P_{p-m}(\cos\alpha) - P_{p+m}(\cos\alpha)}{2} \sin mt \tag{2.6}$$

Substituting their Fourier series expansions from (2.5) and (2.6) for $g_{k-n}(t)$ and $h_{k-n}(t)$, respectively, in (2.4), we obtain after simple calculations

$$\sum_{m=1}^{\infty} m(\beta_m \cos mt - \alpha_m \sin mt) = \frac{1}{2} \sum_{k=1}^{\infty} \left[\beta_k - (-1)^k \frac{2P}{k}\right] \times$$

$$\times \sum_{n=0}^{k} (-1)^n C_n P_{k-n}(\cos\alpha) + \frac{P}{\pi} \sum_{m=1}^{\infty} \frac{P_m(\cos\alpha) + P_{-m}(\cos\alpha)}{2} \cos mt +$$

$$+ 2\lambda\gamma \sum_{m=1}^{\infty} \frac{P_m(\cos\alpha) - P_{-m}(\cos\alpha)}{2} \sin mt -$$

$$-\lambda \left\{ \sum_{m=1}^{\infty} \left[\sum_{k=1}^{\infty} \alpha_k \sum_{n=0}^{k} (-1)^n C_n \frac{P_{k-m-n}(\cos\alpha) - P_{k+m-n}(\cos\alpha)}{2} \right] \sin mt - \right.$$

$$\left. - \sum_{m=1}^{\infty} \left[\sum_{k=1}^{\infty} \left(\beta_k - (-1)^k \frac{2P}{k}\right) \sum_{n=0}^{k} (-1)^n C_n \frac{P_{k+m-n}(\cos\alpha) + P_{k-m-n}(\cos\alpha)}{2} \right] \cos mt \right\}$$

$$(-\pi \leqslant t \leqslant \pi)$$

A comparison of the coefficients of $\cos mt$ and $\sin mt$ on both sides of the last equation results in infinite systems of linear equations

$$\sum_{k=1}^{\infty} \left[\beta_k - (-1)^k \frac{2P}{k}\right] \sum_{n=0}^{k} (-1)^n C_n P_{k-n}(\cos\alpha) = 0 \tag{2.7}$$

$$m\beta_m = \frac{P}{\pi} \frac{P_m(\cos\alpha) + P_{-m}(\cos\alpha)}{2} + \qquad (m = 1, 2, \ldots) \tag{2.8}$$
$$+ \lambda \sum_{k=1}^{\infty} \left[\beta_k - (-1)^k \frac{2P}{k}\right] \sum_{n=0}^{k} (-1)^n C_n \frac{P_{k+m-n}(\cos\alpha) + P_{k-m-n}(\cos\alpha)}{2}$$

$$m\alpha_m = \lambda\gamma [P_{-m}(\cos\alpha) - P_m(\cos\alpha)] + \qquad (m = 1, 2, \ldots) \tag{2.9}$$
$$+ \lambda \sum_{k=1}^{\infty} \alpha_k \sum_{n=0}^{k} (-1)^n C_n \frac{P_{k-m-n}(\cos\alpha) - P_{k+m-n}(\cos\alpha)}{2}$$

It is easy to note that the infinite system (2.8) corresponds to the skew-symmetric part of the contact stress under the elastic laps, and the infinite system (2.9), to the symmetric part of the contact stress.

Let us prove that

$$\sum_{n=0}^{k} (-1)^n C_n P_{k-n}(\cos\alpha) \equiv 0 \qquad \text{for } 0 < \alpha < \pi; \quad k = 1, 2, \ldots \tag{2.10}$$

It will hence result directly that (2.7) will be satisfied identically for any coefficients β_k, and therefore, imposes no constraints on these coefficients.

In order to discover the validity of the identity (2.10), let us note that according to (1.26) and (1.36)

$$\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{\sqrt{\cos s-\cos\alpha}\sin ks}{\sin{}^1/_2(s-t)}\,ds=\frac{1}{\sqrt{2}}\sum_{n=0}^{k}(-1)^n C_n\cos(k-n+{}^1/_2)t$$

$$(k=1,2,\ldots)\quad(-\alpha<t<\alpha)$$

Multiplying both sides of this latter by $\dfrac{2}{\pi\sqrt{\cos t-\cos\alpha}}$ and integrating over t between $-\alpha$ and α, we obtain

$$\frac{1}{\pi^2}\int_{-\alpha}^{\alpha}\sin ksK(t,s)\Big|_{t=-\alpha}^{t=\alpha}ds=\sum_{n=0}^{k}(-1)^n C_n P_{k-n}(\cos\alpha)\qquad(0<\alpha<\pi)$$

$$(k=1,2,\ldots)$$

where the kernel $K(t, s)$ is defined by (1.13) and (1.11). Keeping in mind that

$$K(-\alpha,s)\equiv K(\alpha,s)\equiv 0\quad(-\alpha\leqslant s\leqslant\alpha)$$

we obtain the identity (2.10) from this latter equality.

Let us note that the equality (2.7) expresses the equilibrium condition of the elastic laps. Indeed, by integrating both sides of (1.44) between $-\alpha$ and α and utilizing the boundary conditions (1.8), we arrive at (2.7). On the other hand, if the expansion (1.15) is substituted in place of $\varphi'(t)$ into the same formula (1.44), and taking into account that here $\alpha_0 = P/\pi$, and then both sides are integrated between $-\alpha$ and α, we will obtain

$$2\sum_{k=1}^{\infty}\beta_k\sin ka=P(1-\alpha/\pi)\tag{2.11}$$

It follows from the above, that (2.7) and (2.11) are equivalent. Therefore, (2.11) is also an identity, and imposes no constraints on the unknown coefficients.

Let us now try to satisfy the boundary conditions (1.8) by starting from the expansion of $\varphi(t)$ defined by (1.21). We then obtain (2.11) as well as the following equality:

$$\gamma=\frac{P}{2}-\sum_{k=1}^{\infty}\alpha_k\cos k\alpha\tag{2.12}$$

This latter equality permits the determination of the constant γ if the α_k are known. Therefore, the relationship (2.12) will be some equation from which the constant γ can be determined.

This constant has a simple physical meaning. To clarify this meaning, let us integrate both sides of (1.19) in the interval $(-\pi, \pi)$ and let us take (1.20) into consideration; we hence obtain

$$\gamma=\frac{1}{2\pi}\Big[P(\pi-a)+\int_{-\alpha}^{\alpha}\varphi(t)\,dt\Big]$$

or putting $t = \pi x/l$, $l\alpha/\pi = a$

$$\gamma=\frac{P}{2}\left(1-\frac{a}{l}\right)+\frac{1}{2l}\int_{-a}^{a}\varphi\left(\frac{\pi x}{l}\right)dx\tag{2.13}$$

On the other hand, according to (1.1)

$$\frac{du^{(1)}}{dx}=\varepsilon_x^{(1)}=\frac{\psi(x)}{hE_1}=\frac{\varphi(\pi x/l)}{hE_1}$$

Hence

$$u^{(1)}(x) = C + \frac{1}{hE_1} \int_{-a}^{x} \varphi\left(\frac{\pi s}{l}\right) ds$$

The constant C characterizes the rigid displacement of the system of laps-half-plane. It follows from this last formula that $C = u^{(1)}(-a)$, hence we will have

$$u^{(1)}(a) - u^{(1)}(-a) = \frac{1}{hE_1} \int_{-a}^{a} \varphi\left(\frac{\pi x}{l}\right) dx$$

Substituting herein the expression for the integral from (2.13), we obtain

$$u^{(1)}(a) - u^{(1)}(-a) = \frac{(2\gamma + P)\, l + Pa}{hE_1}$$

We hence conclude that the constant γ determines the displacement of the right end-points of the elastic laps relative to the left ends of these same laps. Its approximate expression will be given in the next section.

Let us turn to an investigation of the infinite systems of linear equations (2.8), (2.9). Let us represent these systems as

$$a_m = \lambda \gamma d_m + \lambda \sum_{k=1}^{\infty} A_{mk} a_k \tag{2.14}$$

$$(m = 1, 2, \ldots)$$

$$b_m = e_m + \lambda \sum_{k=1}^{\infty} B_{mk} b_k \tag{2.15}$$

where

$$A_{mk} = \frac{1}{2k} \sum_{n=0}^{k} (-1)^n C_n \left[P_{k-m-n}(\cos\alpha) - P_{k+m-n}(\cos\alpha)\right] \quad (m, k = 1, 2, \ldots)$$

$$B_{mk} = \frac{1}{2k} \sum_{n=0}^{k} (-1)^n C_n \left[P_{k+m-n}(\cos\alpha) + P_{k-m-n}(\cos\alpha)\right] \quad (m, k = 1, 2, \ldots)$$

$$a_m = m\alpha_m, \quad b_m = m\beta_m, \quad d_m = P_{-m}(\cos\alpha) - P_m(\cos\alpha) \quad (m = 1, 2, \ldots)$$

$$e_m = \frac{P}{2\pi}\left[P_m(\cos\alpha) - P_{-m}(\cos\alpha)\right] - 2P\lambda \sum_{k=1}^{\infty} (-1)^k B_{mk} \quad (m = 1, 2, \ldots)$$

Let us note that the infinite system (2.8) can be represented in a form in which the coefficients $b_k - (-1)^k\, 2P$ in (1.45) for the contact stress, will be the unknowns. An investigation of this infinite system will be no different from an investigation of the system (2.15).

Let us turn first to the infinite system (2.14). To investigate it, let us estimate the sum

$$S_m = \lambda \sum_{k=1}^{\infty} |A_{mk}| \qquad (m = 1, 2, \ldots)$$

We have

$$S_m = \frac{\lambda}{2} \sum_{k=1}^{\infty} \frac{1}{k} \left| \sum_{n=0}^{k} (-1)^n C_n \left[P_{k-m-n}(\cos\alpha) - P_{k+m-n}(\cos\alpha)\right] \right| \leqslant$$

$$\leqslant \frac{\lambda}{2} \left[\sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=0}^{k} |C_n|\, |P_{k-m-n}(\cos\alpha)| + \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=0}^{k} |C_n|\, |P_{k+m-n}(\cos\alpha)| \right] =$$

$$= \frac{\lambda}{2}\Big[\sum_{k=1}^{\infty}\frac{|P_{k+m}(\cos\alpha)|}{k} + \sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{k}|C_n||P_{k+m-n}(\cos\alpha)| +$$

$$+ \sum_{k=1}^{\infty}\frac{|P_{k-m}(\cos\alpha)|}{k} + \sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{k}|C_n||P_{k-m-n}(\cos\alpha)|\Big]$$

Interchanging the orders of summation in the repeated sums, we obtain

$$S_m \leqslant \frac{\lambda}{2}\Big[\sum_{k=1}^{\infty}\frac{|P_{k+m}(\cos\alpha)|}{k} + \sum_{k=1}^{\infty}\frac{|P_{k-m}(\cos\alpha)|}{k} +$$

$$+ \sum_{n=1}^{\infty}|C_n|\sum_{k=n}^{\infty}\frac{|P_{k+m-n}(\cos\alpha)|}{k} + \sum_{n=1}^{\infty}|C_n|\sum_{k=n}^{\infty}\frac{|P_{k-m-n}(\cos\alpha)|}{k}\Big]$$

Introducing the notation

$$T_m(\alpha) = \sum_{k=1}^{\infty}\frac{|P_{k+m}(\cos\alpha)|}{k}, \qquad T_{-m}(\alpha) = \sum_{k=1}^{\infty}\frac{|P_{k-m}(\cos\alpha)|}{k}$$

$$R_m^{(n)}(\alpha) = \sum_{p=1}^{\infty}\frac{|P_{m+p}(\cos\alpha)|}{p+n}, \qquad R_{-m}^{(n)}(\alpha) = \sum_{p=1}^{\infty}\frac{|P_{p-m}(\cos\alpha)|}{p+n} \qquad (m,\ n = 1,\ 2, \ldots)$$

we represent the last inequality as

$$S_m \leqslant \frac{\lambda}{2}\Big[T_m(\alpha) + T_{-m}(\alpha) + \sum_{n=1}^{\infty}|C_n|\,(R_m^{(n)}(\alpha) + R_{-m}^{(n)}(\alpha))\Big] \qquad (2.16)$$

It can be shown that the quantity S_m will satisfy a condition for which the system (2.14) is completely regular.

Let us first estimate each sum in the inequality (2.16) separately. To do this, we use a known result of [7]: the inequality

$$|P_n(\cos\alpha)| < \left(\frac{2}{\pi}\right)^{1/2}\frac{1}{\sqrt{n\sin\alpha}} \qquad (0<\alpha<\pi;\quad n = 1, 2, \ldots)$$

holds. Let us note that the constant $\sqrt{2/\pi}$ cannot be replaced by a lesser one. For simplicity of the computations and of the formulas obtained later, let us replace this constant by unity, thereby asserting that the following inequality holds

$$|P_n(\cos\alpha)| < \frac{1}{\sqrt{n\sin\alpha}} \qquad (0<\alpha<\pi;\ n = 1,\ 2, \ldots) \qquad (2.17)$$

Utilizing this inequality we find

$$T_m(\alpha) < \frac{1}{\sqrt{\sin\alpha}}\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k+m}} < \frac{1}{\sqrt{\sin\alpha}}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}} = \frac{\zeta(3/2)}{\sqrt{\sin\alpha}}$$

Here $\zeta(x)$ $(x > 1)$ is a Riemann function [6, 8].

Thus, the following inequality is valid:

$$T_m(\alpha) < \frac{\zeta(3/2)}{\sqrt{\sin\alpha}} \quad (m = 1,\ 2, \ldots) \qquad (2.18)$$

In order to find a more accurate estimate for the sum T_m (α) let us note that

$$\frac{1}{\sqrt{\sin\alpha}}\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k+m}}<\frac{1}{\sqrt{\sin\alpha}}\left(\frac{1}{\sqrt{m+1}}+\int_1^{\infty}\frac{dx}{x\sqrt{m+x}}\right)$$

Computing this latter integral ([6], formula 2.246), we obtain

$$T_m(\alpha)<\frac{1}{\sqrt{\sin\alpha}}\left(\frac{1}{\sqrt{m+1}}+\frac{1}{\sqrt{m}}\ln\frac{\sqrt{m+1}+\sqrt{m}}{\sqrt{m+1}-\sqrt{m}}\right)\qquad (m=1,\,2,\,\ldots)\quad (2.19)$$

It is easy to show that

$$\lim_{m\to\infty}\frac{1}{\sqrt{m}}\ln\frac{\sqrt{m+1}+\sqrt{m}}{\sqrt{m+1}-\sqrt{m}}=0$$

Therefore

$$\frac{1}{\sqrt{m}}\ln\frac{\sqrt{m+1}+\sqrt{m}}{\sqrt{m+1}-\sqrt{m}}\leqslant A\qquad (m=1,\,2,\,\ldots)$$

After performing elementary calculations, this latter inequality becomes

$$m+1\leqslant {}^1/_2\,(1+\operatorname{ch} A\sqrt{m})$$

Substituting herein the power series expansion of the function $\operatorname{ch} A\sqrt{m}$, we arrive at the inequality

$$\left(\frac{A^2}{4}-1\right)m+\frac{A^4m^2}{2\cdot 4!}+\frac{A^6m^3}{2\cdot 6!}+\ldots\geqslant 0$$

It is hence seen that for nonnegativity of the right side it is sufficient to consider that

$${}^1/_4A^2-1\geqslant 0,\quad \text{or}\quad A\geqslant 2$$

Therefore, the inequality

$$\frac{1}{\sqrt{m}}\ln\frac{\sqrt{m+1}+\sqrt{m}}{\sqrt{m+1}-\sqrt{m}}\leqslant 2\qquad (m=1,\,2,\,\ldots)\qquad (2.20)$$

has been established.

Let us now estimate the sum $T_{-m}(\alpha)$. Using the formula $P_{-n}(\cos\alpha)=P_{n-1}(\cos\alpha)$ known in [6], as well as the equality $P_0(\cos\alpha)=1$, we represent the sum $T_{-m}(\alpha)$ as follows

$$T_{-m}(\alpha)=\sum_{k=1}^{m-2}\frac{|P_{m-k-1}(\cos\alpha)|}{k}+\sum_{k=m+1}^{\infty}\frac{|P_{k-m}(\cos\alpha)|}{k}+\frac{1}{m-1}+\frac{1}{m}\qquad (m\geqslant 3)$$

Let us replace $k-m$ by k in the second sum on the right, and let us then apply the inequality (2.17) to the first two terms, we hence obtain (2.21)

$$T_{-m}(\alpha)<\frac{1}{\sqrt{\sin\alpha}}\left[\sum_{k=1}^{m-2}\frac{1}{k\sqrt{m-k-1}}+\sum_{k=1}^{\infty}\frac{1}{(k+m)\sqrt{k}}\right]+\frac{1}{m-1}+\frac{1}{m}\qquad (m\geqslant 3)$$

Furthermore, let us estimate the sum

$$H_m=\sum_{k=1}^{m-1}\frac{1}{k\sqrt{m-k}}\qquad (m=2,\,3,\,\ldots)$$

To this end, let us note that

$$H_m<\frac{1}{\sqrt{m-1}}+\int_1^{m}\frac{dx}{x\sqrt{m-x}}=\frac{1}{\sqrt{m-1}}+\frac{1}{\sqrt{m}}\ln\frac{\sqrt{m}+\sqrt{m-1}}{\sqrt{m}-\sqrt{m-1}}\qquad (m\geqslant 2)$$

We have for the second sum on the right side of inequality (2.21)

$$\sum_{k=1}^{\infty}\frac{1}{(k+m)\sqrt{k}}<\frac{1}{m+1}+\int_1^{\infty}\frac{dx}{(m+x)\sqrt{x}}$$

or after evaluating the last integral ([6], formula 2.246)

$$\sum_{k=1}^{\infty} \frac{1}{(k+m)\sqrt{k}} < \frac{1}{m+1} + \frac{\pi}{\sqrt{m}} - \frac{2}{\sqrt{m}} \operatorname{arc\,tg} \frac{1}{\sqrt{m}}$$

Using the deduced inequality, we obtain from (2.21)

$$T_{-m}(\alpha) < \frac{1}{\sqrt{\sin\alpha}} \left[\frac{1}{\sqrt{m-1}} \ln \frac{\sqrt{m-1}+\sqrt{m-2}}{\sqrt{m-1}-\sqrt{m-2}} - \frac{2}{\sqrt{m}} \operatorname{arc\,tg} \frac{1}{\sqrt{m}} + \right.$$
$$\left. + \frac{\pi}{\sqrt{m}} + \frac{1}{\sqrt{m-2}} + \frac{1}{m+1}\right] + \frac{1}{m-1} + \frac{1}{m} \qquad (m \geqslant 3) \tag{2.22}$$

Utilizing the inequality (2.20), we obtain from (2.21)

$$T_{-m}(\alpha) < \frac{5}{6} + \frac{3+\zeta(^3/_2)}{\sqrt{\sin\alpha}} \qquad (m=3,4,\ldots)$$

A separate examination of the sums $T_{-1}(\alpha)$ and $T_{-2}(\alpha)$ leads to the inequalities

$$T_{-1}(\alpha) < 1 + \frac{\zeta(^3/_2)}{\sqrt{\sin\alpha}}, \qquad T_{-2}(\alpha) < \frac{3}{2} + \frac{\zeta(^3/_2)}{\sqrt{\sin\alpha}}$$

A comparison of the last three inequalities shows that

$$T_{-m}(\alpha) < \frac{5}{6} + \frac{3+\zeta(^3/_2)}{\sqrt{\sin\alpha}} \qquad (m=1,2,\ldots) \tag{2.23}$$

By methods completely analogous to those elucidated, we obtain the inequalities

$$R_m^{(n)}(\alpha) < \frac{1}{\sqrt{\sin\alpha}} \left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} + \frac{1}{\sqrt{m}} \ln \frac{\sqrt{m+1}+\sqrt{m}}{\sqrt{m+1}-\sqrt{m}}\right) \quad (m, n = 1, 2, \ldots) \tag{2.24}$$

$$R_m^{(n)}(\alpha) < \frac{1}{\sqrt{\sin\alpha}} \left[1 + \zeta\left(\frac{3}{2}\right)\right] \qquad (m, n = 1, 2, \ldots) \tag{2.25}$$

$$R_{-m}^{(n)}(\alpha) < \frac{1}{\sqrt{\sin\alpha}} \left(\frac{1}{\sqrt{m-1}} \ln \frac{\sqrt{m-1}+\sqrt{m-2}}{\sqrt{m-1}-\sqrt{m-2}} - \frac{2}{\sqrt{m}} \operatorname{arc\,tg} \frac{1}{\sqrt{m}} + \right.$$
$$\left. + \frac{2}{\sqrt{m-2}} + \frac{\pi}{\sqrt{m}} + \frac{1}{m+1}\right) + \frac{1}{m-1} + \frac{1}{m} \qquad \left(\begin{matrix} m = 3, 4, \ldots \\ n = 1, 2, \ldots \end{matrix}\right) \tag{2.26}$$

$$R_{-m}^{(n)}(\alpha) < \frac{1}{\sqrt{\sin\alpha}} \left[3 + \zeta\left(\frac{3}{2}\right)\right] + \frac{5}{6} \qquad (m, n = 1, 2, \ldots) \tag{2.27}$$

Finally, by using (2.17) we obtain estimates for the coefficients C_n

$$|C_1| < \frac{1}{\sqrt{\sin\alpha}}, \qquad |C_2| < \frac{1}{3}\left(1 + \frac{1}{\sqrt{2\sin\alpha}}\right), \qquad |C_n| < \frac{1}{\sqrt{\sin\alpha}} \frac{1}{(n-2)^{3/2}}$$
$$(n = 3, 4, \ldots)$$

The latter inequalities permit us to write

$$\sum_{n=1}^{\infty} |C_n| < \frac{1}{\sqrt{\sin\alpha}} + \frac{1}{3}\left(1 + \frac{1}{\sqrt{2\sin\alpha}}\right) + \frac{\zeta(^3/_2)}{\sqrt{\sin\alpha}} \tag{2.28}$$

Taking account of (2.18), (2.23), (2.25), (2.27) and (2.28) we find from (2.16)

$$S_m < \frac{\lambda}{\sin\alpha} \frac{1}{36} \left[\frac{29}{\sqrt{2}} + 131 + \left(171 + \frac{12}{\sqrt{2}}\right) \zeta\left(\frac{3}{2}\right) + 36\zeta^2\left(\frac{3}{2}\right)\right] (m = 1, 2, \ldots)$$

In order for the infinite system of linear equations (2.14) to be completely regular, it is necessary to satisfy the conditions [9]

$$S_m \leqslant \theta < 1 \qquad (m = 1, 2, \ldots)$$

which results in the infinite system (2.14) being completely regular for values of λ satisfying the condition

$$\frac{\lambda}{\sin\alpha} < 36\left[\frac{29}{\sqrt{2}} + 131 + \left(171 + \frac{12}{\sqrt{2}}\right)\zeta\left(\frac{3}{2}\right)\zeta^2\left(\frac{3}{2}\right)\right]^{-1}$$

Substituting $\zeta\,(^3/_2) = 2.612$ [8] in the right side of this inequality, we represent it after simple manipulations as

$$\frac{\lambda}{\sin\alpha} < \frac{1}{25} \tag{2.29}$$

Therefore, the infinite system of linear equations (2.14) is completely regular under the condition (2.29).

The proof that the infinite system of linear equations (2.15) is also completely regular under the same condition (2.29) is no different than that presented.

For other values of the parameter λ the infinite systems (2.14) and (2.15) are quasi-completely regular. Indeed, after some simplification of the expressions in the right sides of the inequalities (2.19), (2.22), (2.24), (2.26) and (2.28), the inequality (2.16) becomes

$$S_m < \frac{3\lambda}{\sqrt{\sin\alpha}}\, G_m \qquad (m = 3, 4, \ldots) \tag{2.30}$$

$$G_m = \frac{2}{\sqrt{m-1}} \ln\frac{\sqrt{m+1} + \sqrt{m}}{\sqrt{m+1} - \sqrt{m}} + \frac{4}{\sqrt{m-2}} + \frac{\pi}{\sqrt{m}} + \frac{3}{m-1} \tag{2.31}$$

Since $\lim G_m = 0$ as $m \to \infty$, the right side of (2.30) can be made arbitrarily small for any λ for sufficiently large m. This circumstance certainly permits the assertion that both infinite systems of linear equations (2.14) and (2.15) are quasi-completely regular for

$$\frac{\lambda}{\sin\alpha} \geqslant \frac{1}{25}$$

where a number N can be given exactly for which the mentioned systems start to be completely regular.

Since the forcing terms of the infinite systems of linear equations (2.14) and (2.15) are bounded as a set, or more accurately, tend to zero as the velocity $O\,(m^{-1/2})$, then according to the theory of completely regular and quasi-completely regular systems [9], they have unique solutions in the class of bounded sequences. These solutions can be obtained by successive approximations, by starting from any bounded initial values in the set. They can also be obtained by the method of reduction. After the a_k and b_k have been determined, the coefficients α_k and β_k can be found by means of the formulas

$$\alpha_k = \frac{a_k}{k}, \qquad \beta_k = \frac{b_k}{k}$$

If $a_k^{(n)}$ and $b_k^{(n)}$ are approximate values of the coefficients a_k and b_k which approach them as $n \to \infty$ then the approximate values $\alpha_k^{(n)}$ and $\beta_k^{(n)}$ of the coefficients α_k and β_k can be determined from the formulas

$$\alpha_k^{(n)} = \frac{a_k^{(n)}}{k}, \qquad \beta_k^{(n)} = \frac{b_k^{(n)}}{k}$$

Let us note that $a_k^{(n)}$ and $b_k^{(n)}$ can be successive approximations or solutions of the truncated systems of a finite number of linear equations when the method of reduction is applied to the infinite systems (2.14) and (2.15).

In conclusion, let us note that more exact estimates could be obtained. However, this would complicate the structure of the final formulas.

3. Investigation of the state of stress of the elastic laps. We precede the investigation of the state of stress of elastic laps by elucidating some results which will permit, on the one hand, giving a foundation to the formal operations performed above, and on the other, verification of the validity of the analytical Fourier series apparatus utilized here. First, let us explain the question of the convergence of the trigonometric series encountered in the previous sections. Second, let us examine the following important question. The coefficients α_k and β_k which are used to form the trigonometric series for the functions $\varphi'(t)$ and $\varphi\ (t)$ are determined from infinite systems of linear equations. There is no advance guarantee that the coefficients α_k and β_k thus determined will be Fourier coefficients of some function with specific properties. Questions of the existence and determination of functions having a previously assigned sequence of numbers as the sequence of Fourier coefficients are among the known trigonometric problems of moments. However, existing criteria from this area are unsuitable in practice since they do not permit any verification in specific cases as to whether for a given sequence of numbers such a function is of a definite class which would have this sequence of numbers as its Fourier coefficients. The clarification of these questions for the considered trigonometric series turns out to be elementary and based on the estimates of their coefficients.

Let us proceed to estimate the coefficients. To do this, we write the infinite systems (2.14), (2.15) as

$$\gamma_m = \lambda\gamma d_m m^{1/2-\delta} + \lambda \sum_{k=1}^{\infty} A_{mk}\left(\frac{m}{k}\right)^{1/2-\delta}\gamma_k \tag{3.1}$$

$$\delta_m = e_m m^{1/2-\delta} + \lambda \sum_{k=1}^{\infty} B_{mk}\left(\frac{m}{k}\right)^{1/2-\delta}\delta_k \tag{3.2}$$

$$\gamma_m = a_m m^{1/2-\delta} = m^{3/2-\delta}\alpha_m, \qquad \delta_m = b_m m^{1/2-\delta} = m^{3/2-\delta}\beta_m \qquad (m=1, 2, \ldots)$$

and δ is an arbitrarily small positive number. Let us estimate the sums

$$V_m = \lambda \sum_{k=1}^{\infty} |A_{mk}|\left(\frac{m}{k}\right)^{1/2-\delta} \qquad (m=1, 2, \ldots)$$

We have

$$V_m < \lambda m^{1/2-\delta} \sum_{k=1}^{\infty} |A_{mk}|$$

Taking into account (2.30), we can write

$$V_m < \frac{3\lambda m^{1/2-\delta}}{\sqrt{\sin\alpha}} G_m \qquad (m=3, 4, \ldots). \tag{3.3}$$

Using L'Hopital's rule, it is easy to show that

$$\lim_{m\to\infty} \frac{1}{m^\delta} \ln \frac{\sqrt{m+1}+\sqrt{m}}{\sqrt{m+1}-\sqrt{m}} = 0 \tag{3.4}$$

Taking account of the last equality and using the expression for G_m from (2.31), we conclude that

$$\lim V_m = 0 \quad \text{for } m \to \infty \tag{3.5}$$

In order to obtain an estimate independent of m for V_m, let us note that because of (3.4)

$$\frac{1}{m^\delta} \ln \frac{\sqrt{m+1}+\sqrt{m}}{\sqrt{m+1}-\sqrt{m}} \leqslant K \qquad (m=1, 2, \ldots)$$

Hence, as in the derivation of the inequality (2.20), we obtain

$$m \leqslant \frac{K^2 m^{2\delta}}{2 \cdot 2} + \frac{K^4 m^{4\delta}}{2 \cdot 4!} + \ldots + \frac{K^{2k} m^{2k\delta}}{2 \cdot (2k)!} + \ldots \tag{3.6}$$

It is easy to see that

$$\frac{K^{2k} m^{2k\delta}}{2 \cdot (2k)!} > \frac{K^{2k} m^{k/q}}{2 \cdot (2k)!} \qquad (m = 1, 2, \ldots), \qquad q = E(\delta^{-1})$$

Here $E(x)$ is the integer part of the number x. Let us select k so that

$$\frac{K^{2k} m^{k/q}}{2 \cdot (2k)!} > m \qquad (m = 1, 2, \ldots)$$

It is evidently sufficient to consider $k = q$. Therefore, if

$$K \geqslant [2 \cdot (2q)!]^{1/2q}$$

then the inequality (3.6) is satisfied. Therefore

$$\frac{1}{m^\delta} \ln \frac{\sqrt{m+1} + \sqrt{m}}{\sqrt{m+1} - \sqrt{m}} \leqslant L = [2 \cdot (2q)!]^{1/2q} \qquad (m = 1, 2, \ldots) \tag{3.7}$$

Utilizing the inequality (3.7), we find by using (2.31) (*)

$$V_m < \lambda V / \sqrt{\sin \alpha}, \quad V = 3(L\sqrt{6} + 4\sqrt{3} + \pi + {}^3/_2\sqrt{3}) \quad (m = 1, 2, \ldots) \tag{3.8}$$

Putting

$$\theta = \frac{\lambda V}{\sqrt{\sin \alpha}} \tag{3.9}$$

we find that $\theta < 1$ if λ satisfies the condition

$$\frac{\lambda}{\sqrt{\sin \alpha}} < \frac{1}{V} \tag{3.10}$$

Therefore, if condition (3.10) holds, then the infinite system (3.1) is completely regular, but in the opposite case it is quasi-completely regular, as follows from (3.5). This assertion is evidently valid for the infinite system (3.2) also.

Furthermore, it is easy to find that

$$|\gamma_m| \leqslant \lambda |\gamma| m^{1/2-\delta} |d_m| + \lambda M \sum_{k=1}^{\infty} |A_{mk}| \left(\frac{m}{k}\right)^{1/2-\delta} <$$

$$< \lambda |\gamma| m^{1/2-\delta} |d_m| + M V_m \qquad (m = 1, 2, \ldots) \tag{3.11}$$

The fact that $|\gamma_k| \leqslant M$ $(k = 1, 2, \ldots)$ has been utilized in deducing this inequality since according to what has just been proved, the infinite system (3.1) of linear equations is completely regular or quasi-completely regular, and therefore, has a unique bounded solution.

Taking into consideration the inequalities (2.17) and (3.5), we find from (3.11) that $\lim \gamma_m = 0$ for $m \to \infty$. Analogously, it can be shown also that $\lim \delta_m = 0$ as $m \to \infty$.

This latter circumstance permits the assertion that

$$\alpha_m = o\left(\frac{1}{m^{3/2-\delta}}\right), \qquad \beta_m = o\left(\frac{1}{m^{3/2-\delta}}\right)$$

This means the following formulas also hold

$$\alpha_m = O\left(\frac{1}{m^{3/2-\delta}}\right), \qquad \beta_m = O\left(\frac{1}{m^{3/2-\delta}}\right) \tag{3.12}$$

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*) For $m \geqslant 3$ this inequality is obtained from (3.3). But it also holds for $m = 1, 2$.

Since the coefficients α_m and β_m are quantities of the mentioned orders, the trigonometric series in (1.19) converges absolutely and uniformly. There results from just the uniform convergence that this trigonometric series is a Fourier series of some function $F(t) - {}^1/_2\, Pt/\pi$. We denote it by $\varphi(t) - {}^1/_2\, Pt/\pi$ in the interval $(-\alpha, \alpha)$. Therefore, the function $\varphi(t)$ $(-\alpha \leqslant t \leqslant \alpha)$, continued into the interval $(-\pi, \pi)$ by means of (1.20), is represented by the Fourier series (1.22).

Let us investigate the convergence of the trigonometric series obtained by formal differentiation of the Fourier series (1.22), i.e. the series in the right side of (1.15).

The coefficients α_k and β_k of this series are determined, as has been shown, from the infinite systems of linear equations (2.8), (2.9), whose derivation is based on (2.1). Therefore, convergence of the trigonometric series (1.15) is equivalent to convergence of the series in the right side of (2.1).

Furthermore, let us introduce the notation

$$\Phi(t) = \sum_{k=1}^{\infty} \alpha_k \sum_{n=0}^{k} (-1)^n C_n \sin(k - n + {}^1/_2)\, t - \tag{3.13}$$

$$- \sum_{k=1}^{\infty} \beta_k \sum_{n=0}^{k} (-1)^n C_n \cos(k - n + {}^1/_2)\, t \quad (-\alpha \leqslant t \leqslant \alpha)$$

$$\Psi(t) = 2P \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{n=0}^{k} (-1)^n C_n \cos(k - n + {}^1/_2) t \tag{3.14}$$

Taking (3.12) and (2.28) into consideration, we discover that the series (3.13) converges absolutely and uniformly in the segment $[-\alpha, \alpha]$ and therefore its sum $\Phi(t)$ is a continuous function in this same segment.

Returning to the series (3.14), let us intergange the orders of summation, and then let us use the known formulas [6]

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sin kt}{k} = -\frac{t}{2}, \qquad \sum_{k=1}^{\infty} \frac{(-1)^k \cos kt}{k} = \ln \frac{1}{2\cos {}^1/_2 t} \quad (-\pi < t < \pi) \tag{3.15}$$

After having performed elementary calculations, (3.14) becomes

$$\Psi(t) = 2P \cos\frac{t}{2}\left\{\ln\frac{1}{2\cos {}^1/_2 t} + \sum_{n=1}^{\infty} (-1)^n C_n \left[\cos nt \sum_{k=n}^{\infty} \frac{(-1)^k \cos kt}{k} + \right.\right.$$

$$\left.\left. + \sin nt \sum_{k=n}^{\infty} \frac{(-1)^k \sin kt}{k}\right]\right\} + 2P \sin\frac{t}{2}\left\{\frac{t}{2} - \right. \tag{3.16}$$

$$\left. - \sum_{n=1}^{\infty} (-1)^n C_n \left[\cos nt \sum_{k=n}^{\infty} \frac{(-1)^k \sin kt}{k} - \sin nt \sum_{k=n}^{\infty} \frac{(-1)^k \cos kt}{k}\right]\right\}$$

Since the number sequence $u_k = (-1)^k k^{-1}$ will be a sequence of bounded variation, i.e. $\quad |\Delta u_1| + |\Delta u_2| + |\Delta u_3| + \ldots < \infty \quad (\Delta u_k = u_k - u_{k-1})$

Hence, the series (3.15) converges uniformly [10] for $\varepsilon < |t| < \pi$ (ε is an arbitrarily small positive number). For $t = 0$ they evidently simply converge. It follows from the above that

$$|q_n(t)| \leqslant K, \qquad |r_n(t)| \leqslant K \qquad (0 \leqslant t \leqslant \pi)$$

$$q_n(t) = \sum_{k=n}^{\infty} \frac{(-1)^k \cos kt}{k}, \qquad r_n(t) = \sum_{k=n}^{\infty} \frac{(-1)^k \sin kt}{k}$$

On the basis of the last inequalities and inequalities (2.28) we see that a series of the form

$$\sum_{n=1}^{\infty} (-1)^n C_n \begin{matrix} \cos nt \\ \sin nt \end{matrix} \sum_{k=n}^{\infty} \frac{(-1)^k}{k} \begin{matrix} \cos kt \\ \sin kt \end{matrix}$$

converges absolutely and uniformly in the interval $(-\pi, \pi)$, and therefore, also in the segment $-\alpha \leqslant t \leqslant \alpha$. Hence, the function $\Psi(t)$ is continuous in the segment $[-\alpha, \alpha]$.

Therefore, it has been proved that

$$\varphi'(t) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} k(\beta_k \cos kt - \alpha_k \sin kt) = \frac{\chi(t)}{\sqrt{2(\cos t - \cos\alpha)}} \quad (-\alpha < t < \alpha) \tag{3.17}$$

Here the function

$$\chi(t) = P\pi^{-1} \cos {}^1/_2 t - 2\lambda\gamma \sin {}^1/_2 t - \lambda\, [\Phi(t) - \Psi(t)]$$

will be a continuous function in the segment $[-\alpha, \alpha]$.

It follows from (3.17) that the function $\varphi'(t)$ is absolutely integrable in the interval $(-\alpha, \alpha)$. This yields a foundation for the assertion that $F(t)$ from (1.18) is an absolutely continuous function in the interval $(-\pi, \pi)$. But according to (1.20), $F(t) \equiv \varphi(t)$ for $-\alpha \leqslant t \leqslant \alpha$. Therefore, the function $\varphi(t)$ $(-\alpha \leqslant t \leqslant \alpha)$ will also be an absolutely continuous function represented by the Fourier series (1.22). There remains to utilize the following results of [10] to see that the series (1.15) is a Fourier series (*) for $\varphi'(t)$. The trigonometric series obtained by formal differentiation of the Fourier series of an absolutely continuous function is the Fourier series for its derivative. Therefore, a foundation has been given for the Fourier series of the function $\varphi(t)$ from (1.22) to be differentiated term by term, and for a Fourier series representing the function $\varphi'(t)$ to be obtained again.

Turning to the investigation of the state of stress of the elastic laps, we recall that the contact stresses are determined by (1.44), or more exactly, by (1.45). This formula has been represented in the form (3.17) with clearly isolated singularities which characterize the state of stress of the elastic laps near their ends. It is seen from this formula that the contact stresses at the ends of the elastic laps become of integrable order at infinity. Simultaneously, the presented analyses permit the assertion that the law of contact stress distribution under periodically repeated laps has an analytical structure such that the singularities inherent in the state of stress of the elastic laps near their ends are of the same kind as in the case of periodically repeated rigid stamps. Therefore, the assertion that singularities characterizing the state of stress of the elastic laps near their ends are explicitly extracted in (1.44), which has been expressed in Sect. 1, has now been given a complete foundation.

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*) More exactly, we speak here of a Fourier series for the function $f(t)$ defined by (1.6). which agrees with the function $\varphi'(t)$ in the interval $(-\alpha, \alpha)$. This remark also refers to the functions $F(t)$ and $\varphi(t)$.

Let us note the following. The coefficients α_k and β_k in (1.44) and (3.17) are determined by successive approximations or the method of reduction from the infinite systems (2.8) and (2.9), represented in the form (2.14), (2.15), or (3.1), (3.2). If the successive approximations $\alpha_k^{(n)}$ and $\beta_k^{(n)}$ which converge to α_k and β_k, respectively, as $n \to \infty$, are substituted in (1.44) or (3.17) in place of α_k and β_k, then a certain functional sequence is obtained

$$\chi_n(t) = \frac{P\pi^{-1}\cos{}^1/_2\, t - 2\lambda\gamma \sin{}^1/_2\, t - \lambda\,[\Phi_n(t) - \Psi(t)]}{\sqrt{2(\cos t - \cos\alpha)}} \qquad (-\alpha < t < \alpha) \tag{3.18}$$

$$\Phi_n(t) = \sum_{k=1}^{\infty} \alpha_k^{(n)} \sum_{k=0}^{k} (-1)^n C_n \sin(k - n + {}^1/_2)t -$$
$$- \sum_{k=1}^{\infty} \beta_k^{(n)} \sum_{n=0}^{k} (-1)^n C_n \cos(k - n + {}^1/_2)t \quad (n = 1, 2, \ldots) \tag{3.19}$$

Let us prove that the sequence $\chi_n(t)$ tends uniformly to the function $\varphi'(t)$ as $n \to \infty$. It is easy to see that such a proof will reduce to the proof of uniform convergence of the sequence $\Phi_n(t)$ to $\Phi(t)$ as $n \to \infty$.

Proceeding to this latter proof, we note that we can put

$$\alpha_k^{(n)} = \frac{\gamma_k^{(n)}}{k^{3/2-\delta}}, \qquad \beta_k^{(n)} = \frac{\delta_k^{(n)}}{k^{3/2-\delta}} \tag{3.20}$$

where $\gamma_k^{(n)}$ and $\delta_k^{(n)}$ are the successive approximations which tend to the solutions γ_k and δ_k of the infinite systems of linear equations (3.1), (3.2) as $n \to \infty$.

We have

$$|\Phi_n(t) - \Phi(t)| \leqslant \sum_{k=1}^{\infty} \frac{|\gamma_k^{(n)} - \gamma_k|}{k^{3/2-\delta}} \sum_{n=0}^{k} |C_n| + \sum_{k=1}^{\infty} \frac{|\delta_k^{(n)} - \delta_k|}{k^{3/2-\delta}} \sum_{n=0}^{k} |C_n|$$
$$(-\alpha \leqslant t \leqslant \alpha) \tag{3.21}$$

Furthermore, it is necessary to obtain an estimate for $|\gamma_k^{(n)} - \gamma_k|$ and $|\delta_k^{(n)} - \delta_k|$. To do this, let us use the Banach principle of contracted mappings [11]. This principle permits establishment not only of complete regularity, or quasi-complete regularity of the infinite systems of linear equations (2.14), (2.15) or (3.1), (3.2), but also the estimates we need. In this connection, let us present some elementary information from functional analysis [11].

Let us introduce the set Ξ of all bounded number sequences $x = \{\xi_1, \xi_2, \ldots\}$. This means that $|\xi_i| \leqslant K_x$ for all i, where K_x is a constant dependent only on the element x. Let $x = \{\xi_i\}$ and $y = \{\eta_i\}$ belong to Ξ. Let us introduce a metric by means of the equality

$$\rho(x, y) = \sup_i \{\xi_i - \eta_i\}$$

As is known [11], the set Ξ with the metric introduced by such an equality becomes a complete metric space. It is called the space m of bounded number sequences.

Let us consider the linear operator $y = Ax$ in the space m which has been given by using the equalities

$$\eta_i = \sum_{k=1}^{\infty} a_{ik}\xi_k + b_i \qquad (i = 1, 2, \ldots), \qquad \{b_i\} \in m \tag{3.22}$$

Let us assume that the infinite matrix $\|a_{ik}\|_{i,k=1}^{\infty}$ is such that $\{\xi_i\} \in m$, then $\{\eta_i\} \in m$

also, i. e. the operator A transforms an element of the space m again into an element of the same space. We have

$$\rho(y_1, y_2) = \rho(Ax_1, Ax_2) = \sup_i |\eta_i^{(1)} - \eta_i^{(2)}| =$$

$$= \sup_i \left| \sum_{k=1}^{\infty} a_{ik} (\xi_k^{(1)} - \xi_k^{(2)}) \right| \leqslant \sup_i \sum_{k=1}^{\infty} |a_{ik}| \, |\xi_k^{(1)} - \xi_k^{(2)}| \leqslant$$

$$\leqslant \sup_i \sum_{k=1}^{\infty} |a_{ik}| \sup_k |\xi_k^{(1)} - \xi_k^{(2)}| = \sup_i \sum_{k=1}^{\infty} |a_{ik}| \rho(x_1, x_2)$$

Let us assume that

$$\sum_{k=1}^{\infty} |a_{ik}| \leqslant \theta < 1 \tag{3.23}$$

for all i. If condition (3.23) is satisfied, then according to the Banch principle of contracted mappings it can be asserted that the operator A, given by (3.22), has a single fixed point x_0 in the space m such that $Ax_0 = x_0$. The point x_0 can be found by successive approximations, starting from any initial element X. These successive approximations converge to x_0 in the metric of the space m. In other words, upon compliance with condition (3.23), the infinite system of linear equations

$$\xi_i = \sum_{k=1}^{\infty} a_{ik} \xi_k + b_i \tag{3.24}$$

in the space of bounded number sequences has a unique solution $\{\xi_i\}$, which can be obtained by successive approximations starting from any initial element from the same space; in other words, the system (3.24) is completely regular.

Let us note that an estimate of the closeness of the nth approximation x_n to the fixed point x_0 is hence given by the formula

$$\rho(x_n, x_0) = \frac{\theta^n}{1-\theta} \rho(X, AX) \tag{3.25}$$

In case the conditions

$$\sum_{k=1}^{\infty} |a_{ik}| \leqslant \theta < 1 \quad (i = N+1, \ldots), \qquad \sum_{k=1}^{\infty} |a_{ik}| < \infty \quad (i = 1, 2, \ldots, N)$$

are satisfied instead of condition (3.23), the operator A defined above can be examined in the subspace R_N of the space m, which consists of element whose first N components are zero. In this case, again on the basis of the Banach principle, it can be asserted that the operator A has a single fixed point in the subspace R_N, i. e. the infinite system (3.24) is quasi-completely regular.

Therefore, the classical theory of completely regular and quasi-completely regular infinite systems of linear equations is included in the general scheme representing the Banach contracted mapping principle.

Now, let us utilize the estimate (3.25) to the infinite systems (3.1), (3.2). We have

$$\rho(x_n, x_0) = \frac{\theta^n}{1-\theta} \rho(X, A_1 X), \qquad \rho(y_n, y_0) = \frac{\theta^n}{1-\theta} \rho(X, A_2 X),$$

where

$$x_n = \{\gamma_k^{(n)}\}, \qquad x_0 = \{\gamma_k\}, \qquad y_0 = \{\delta_k^{(n)}\}, \qquad y_0 = \{\delta_k\}$$

and A_1 and A_2 are operators corresponding, in the above-mentioned sense, to the infinite systems of linear equations (3.1), (3.2), respectively.

Returning to (3.21), we find

$$|\Phi_n(t)-\Phi(t)|\leqslant\frac{2\theta^n\mu}{1-\theta}\sum_{k=1}^{\infty}\frac{1}{k^{3/2-\delta}}\sum_{n=0}^{k}|C_n| \qquad (-\alpha\leqslant t\leqslant\alpha) \tag{3.26}$$

where

$$\mu=\max(\mu_1,\mu_2),\quad \mu_1=\rho(X, A_1X),\quad \mu_2=\rho(X, A_2X) \tag{3.27}$$

and θ is defined by (3.9).

Using the inequality (2.28), we obtain from (3.26)

$$|\Phi_n(t)-\Phi(t)|<\frac{2\theta^n\mu}{1-\theta}\,\frac{5+3\zeta(3/2)}{3\sqrt{\sin\alpha}}\,\zeta\left(\frac{3}{2}-\delta\right) \qquad (-\alpha\leqslant t\leqslant\alpha) \tag{3.28}$$

The inequality (3.28) shows the validity of the above-mentioned assertions.

Therefore, for λ satisfying condition (3.10), the sequence $\chi_n(t)$ tends to the function $\varphi'(t)$ uniformly in $t\ (-\alpha<t<\alpha)$ as $n\to\infty$. It is easy to show that this assertion holds even when λ does not satisfy the condition (3.10). However, we shall not consider this point.

Let us examine the expression

$$|\chi_n(t)-\varphi'(t)|=\frac{\lambda\,|\Phi_n(t)-\Phi(t)|}{\sqrt{2(\cos t-\cos\alpha)}} \qquad (-\alpha<t<\alpha)$$

which is the absolute difference between the approximate and true expressions for the contact stresses under the elastic laps. In order to estimate this difference, let us note that the function $[2(\cos t-\cos\alpha)]^{-1/2}$ defines those singularities which are inherent to the contact stresses near the ends of the elastic laps. It is hence natural to judge the closeness of the approximate expressions $\chi_n(t)$ of the contact stresses to the true $\varphi'(t)$ by estimating the difference $|\Phi_n(t)-\Phi(t)|$. We can write this estimate, represented by (3.28), as (*)

$$\sup_{t\in(-\alpha,\alpha)}|\Phi_n(t)-\Phi(t)|<\frac{2\theta^n\mu}{1-\theta}\,\frac{5\zeta(3/2)+3\zeta^2(3/2)}{3\sqrt{\sin\alpha}} \tag{3.29}$$

Let us now borrow the estimates μ_1 and μ_2 from (3.27). It is known that the selection of the element X is arbitrary and affects only the rapidity of convergence of the successive approximations. Let us take the null element as X for the case of the operator A_1 or the infinite system (3.1). Then

$$\mu_1=\sup_m\{\lambda|\gamma|\,m^{1/2-\delta}|P_{-m}(\cos\alpha)-P_m(\cos\alpha)|\}\leqslant$$
$$\leqslant\lambda|\gamma|\{\sup_m m^{1/2-\delta}|P_{-m}(\cos\alpha)|+\sup_m m^{1/2-\delta}|P_m(\cos\alpha)|\}<$$
$$<\frac{\lambda|\gamma|}{\sqrt{\sin\alpha}}\left(\sup_{m=2,3,\ldots}\left(\frac{m}{m-1}\right)^{1/2}+1\right)<\frac{\lambda|\gamma|(1+\sqrt{2})}{\sqrt{\sin\alpha}}$$

We take $X=\{(-1)^k 2P\}$ in the case of the operator A_2 or the infinite system (3.2), we obtain

$$\mu_2=\frac{P}{2\pi}\sup_m[m^{1/2-\delta}|P_m(\cos\alpha)+P_{-m}(\cos\alpha)|<\frac{P(1+\sqrt{2})}{2\pi\sqrt{\sin\alpha}}$$

Therefore, the following estimates hold:

$$\mu_1<\frac{\lambda|\gamma|(1+\sqrt{2})}{\sqrt{\sin\alpha}},\qquad \mu_2<\frac{P(1+\sqrt{2})}{2\pi\sqrt{\sin\alpha}} \tag{3.30}$$

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*) In practice, we can consider $\delta=0$.

Hence, it is seen that the constants μ_1 and μ_2, and therefore, also μ, depend on the quantities λ, $|\gamma|$ and P.

Considering the nth approximations $\{\gamma_k^{(n)}\}$ and $\{\delta_k^{(n)}\}$, which are constructed from the infinite systems of linear eqautions (3.1), (3.2), respectively, by starting from the above-mentioned initial elements X, we find $\{\alpha_k^{(n)}\}$ and $\{\beta_k^{(n)}\}$ from (3.20). We then form the functions $\chi_n(t)$ and $\Phi_n(t)$ by means of (3.18), (3.19). These functions define the approximate expressions for the contact stresses. How close they are to the true expression for the contact stresses is seen from the inequality (3.29). This inequality shows that $\sup|\Phi_n(t) - \Phi(t)|$ becomes less and less for $t \in (-\alpha, \alpha)$ as the number n of the successive approximations increases.

Let us consider the first approximation in rather more detail. We find for the first approximations $\{\gamma_k^{(1)}\}$ and $\{\delta_k^{(1)}\}$ from the infinite systems (3.1), (3.2)

$$\gamma_k^{(1)} = \lambda\gamma k^{1/2-\delta}\,[P_{-k}(\cos\alpha) - P_k(\cos\alpha)]$$
$$\delta_k^{(1)} = \frac{P}{2\pi} k^{1/2-\delta}\,[P_k(\cos\alpha) + P_{-k}(\cos\alpha)] \qquad (k = 1, 2, \ldots)$$

and therefore, according to (3.20)

$$\alpha_k^{(1)} = \lambda\gamma\frac{P_{-k}(\cos\alpha) - P_{-k}(\cos\alpha)}{k}, \quad \beta_k^{(1)} = \frac{P}{2\pi}\frac{P_k(\cos\alpha) + P_{-k}(\cos\alpha)}{k} \quad (k = 1, 2, \ldots)$$

From (2.12) we find the approximate value (in a first approximation) of the constant γ

$$\gamma \approx P\left[2 - 2\lambda\sum_{k=1}^{\infty}\frac{P_{-k}(\cos\alpha) - P_k(\cos\alpha)}{k}\cos k\alpha\right]^{-1} \tag{3.31}$$

Therefore, the contact stresses under the elastic laps, determined by (1.45), can be evaluated in a first approximation by means of the computational formula

$$\begin{aligned}\tau(x) = {} & \frac{P\cos(\pi x/2l)}{l\sqrt{2[\cos(\pi x/l) - \cos(\pi a/l)]}} - \frac{2\pi\lambda\gamma\sin(\pi x/2l)}{l\sqrt{2[\cos(\pi x/l) - \cos(\pi a/l)]}} - \\ & - \frac{\lambda\pi}{l\sqrt{2[\cos(\pi x/l) - \cos(\pi a/l)]}}\left\{\sum_{k=1}^{\infty}\lambda\gamma\frac{P_{-k}(\cos\alpha) - P_k(\cos\alpha)}{k}\times\right. \\ & \times\sum_{n=0}^{k}(-1)^n C_n \sin\left[(k-n+1/2)\frac{\pi x}{l}\right] - \\ & - \sum_{k=1}^{\infty}\left[\frac{P}{2\pi}\frac{P_k(\cos\alpha) + P_{-k}(\cos\alpha)}{k} - (-1)^k\frac{2P}{k}\right]\times \\ & \left.\times\sum_{n=1}^{k}(-1)\,C_n\cos\left[(k-n+1/2)\frac{\pi x}{l}\right]\right\} \qquad (-a < x < a)\end{aligned} \tag{3.32}$$

where γ and C_n are determined, respectively, from (3.31) and (1.43).

The absolute error admitted here is determined according to (3.29) by the inequality

$$\sup_{t\in(-\alpha,\alpha)}|\Phi_1(t) - \Phi(t)| < \frac{2\theta\mu}{1-\theta}\,\frac{5\zeta(3/2) + 3\zeta^2(3/2)}{3\sqrt{\sin\alpha}} \tag{3.33}$$

where θ is defined by (3.9), and μ by (3.30). The left side of the inequality (3.33) is evidently arbitrarily small for sufficiently small λ.

Appropriate estimates can also be obtained when λ does not satisfy the inequality

(3.10), and therefore, the infinite systems of linear equations (3.1), (3.2) are quasi-completely regular.

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DUAL TRIGONOMETRIC SERIES IN CRACK AND PUNCH PROBLEMS

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B. A. KUDRIAVTSEV and V. Z. PARTON
(Moscow)

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The author obtains the solution of a certain class of dual trigonometric series with the aid of a method proposed by Tranter [1]. Certain crack and punch problems, both static and dynamic, reduce to this class. As an example the problem of steady-state vibration of an unbounded plane with a periodic system of slits along the real axis is considered. The solution which is obtained permits the determination of a purely inertial effect which lowers the fracture load.

1. Let us consider the dual trigonometric series

$$\sum_{n=1}^{\infty} nB_n^* \cos n\xi = f(\xi) \qquad (0 \leqslant \xi \leqslant \xi_0)$$
$$\sum_{n=1}^{\infty} B_n^* \cos n\xi = 0 \qquad (\xi_0 \leqslant \xi \leqslant \pi) \tag{1.1}$$